Matemática Contemporânea, Vol. 45, 143–153

On equitable total coloring of complete r-partite graphs

A. G. da Silva **S. Dantas** **D. Sasaki**

Abstract

In 2002, Wang conjectured that the equitable total chromatic number of a graph is either $\Delta+1$ or $\Delta+2$, where Δ is the maximum degree of a graph. In this work, we investigate the equitable total coloring of complete r-partite graphs and verify the conjecture for some of these graphs.

1 Introduction

Let $G = (V, E)$ be a simple connected graph. A *k-total coloring* of G is an assignment of k colors to the vertices and edges of G so that adjacent or incident elements have different colors. The *total chromatic number of* G, denoted by χ'', is the least k for which G has a k-total coloring. The Total Coloring Conjecture (TCC) states that $\Delta + 1 \leq \chi'' \leq \Delta + 2$ [1, 6]. If the difference between the cardinalities of any two color classes is either 0 or 1, then the total coloring is said to be *equitable*. The *equitable total chromatic number of* G, denoted by χ''_e, is the least k for which G has an equitable k-total coloring. Similarly to the total colorings, it was conjectured in

2000 AMS Subject Classification: 68R10, 68R15.

Key Words and Phrases: equitable total coloring, complete r-partite graph, algorithm

Partially supported by CNPq, CAPES, and FAPERJ.

2002 by Wang [7] that $\Delta + 1 \leq \chi''_e \leq \Delta + 2$ (Equitable Total Coloring Conjecture (ETCC)). In 2016, Dantas et al. [3] proved that the problem of determining the equitable total chromatic number of a cubic bipartite graph is NP-complete.

A graph is said to be *r-partite* if there exists a partition of its vertex set $X_1 = \{x_{11}, x_{12}, \cdots, x_{1p_1}\}, X_2 = \{x_{21}, x_{22}, \cdots, x_{2p_2}\}, \cdots, X_r = \{x_{r1}, x_{r2}, \cdots, x_{rp_r}\}$ such that no two vertices within the same part are adjacent. In some cases, we replace the notation x_{ij} by $x_{i,j}$ for the benefit of the reader. A *complete* r-partite graph is an r-partite graph in which there is an edge between any two vertices of different parts of the partition. We denote a complete r-partite graph having p_i, $i = 1, \ldots, r$, vertices in each independent set by $K_{p_1,p_2,\ldots,p_r}$. If $p = p_i = p_j$ for all $1 \leq i < j \leq r$, then the r-partite graph is said to be *p-balanced*. In this case, we denote the graph by $K_{r\times p}$.

We adopt the following convention throughout the text concerning the graph $K_{r\times s}$: the vertices are displayed as in a matrix with r columns and p rows, where each column corresponds to a part X_i of the partition of the vertex set. Therefore, the vertex x_{ij} is the j-th vertex of the part X_i, and it corresponds to the j-th row and in the i-th column.

In 1974, Bermond determined the total chromatic number of all complete r-partite p-balanced graphs [2]. In 1994, Fu [4] investigated the equitable total coloring of complete bipartite graphs and complete r-partite graphs of odd order. In fact, for complete bipartite graphs, Fu proved that $\chi''_e = \chi''$. Furthermore, considering complete r-partite graphs of odd order, Fu proved that there exist equitable $(\Delta + 2)$-total colorings for all of these graphs. In this work, we investigate the equitable total coloring of complete r-partite graphs by verifying the ETCC for the following class of graphs:

1. $K_{2\times p}$, has $\chi''_e = \Delta + 2$;
2. K_{p_1,p_2}, with $p_1 \neq p_2$, has $\chi''_e = \Delta + 1$;
3. $K_{r\times p}$, with r even and p odd, has $\chi''_e = \Delta + 2$;
4. $K_{r\times p}$, with r odd and p even, has $\chi''_e = \Delta + 1$ if $r \geq \frac{p}{2}$.

2 Equitable total coloring of $K_{2\times p}$

Fu [4] determined that $\chi''_e = \Delta + 2$ for $K_{2\times p}$. We provide an algorithm to color such graphs. Edge-coloring matrices are matrices whose entries represent the colors assigned to the edges of a graph. Let $A_{X_1X_2} = [a_{ij}]$ be a $p \times p$ edge-coloring matrix in which the entry a_{ij} represents the color assigned to the edge that has x_{1i} and x_{2j} as its ends. To the entry a_{ij}, assign $i + j - 1 \mod (p)$ if $i + j - 1 \not\equiv 0 \mod (p)$; and p, otherwise. To the vertices of X_1, assign the color $p + 1$ and to the vertices of X_2, assign $p + 2$. For results taken modulo i, if such result is congruent 0 modulo i, then use i instead of 0. See the example when $p = 3$:

$$A_{X_1X_2} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Theorem 1. *The algorithm describes an equitable ($\Delta + 2$)-total coloring of $K_{2\times p}$.*

Ideia of the proof. We need to show that adjacent and incident elements are assigned to different colors and that the difference between the cardinalities of two different color classes is at most 1. Indeed, the vertices of X_1 are nonadjacent, since the graph is bipartite, as well as the vertices of X_2. Since the cardinality of each one of those sets is p, this means that the colors $p+1$ and $p+2$ are used exactly p times each. Since those colors are not used in the edges, no edge receives the same colors as their ends. Moreover, it is possible to prove that every element of $\{1, 2 \cdots, p\}$ appears precisely once in each row (column) and they are not in a conflict. Since $A_{X_1X_2}$ has p rows and each color of $\{1, 2, \cdots, p\}$ appears exactly once per row, we conclude that each one of those p colors are used p times in the coloring of the edges of $K_{2\times p}$. This proves that the difference between the cardinalities of any two distinct color classes is 0. Hence, the total coloring is equitable, as desired. ∎

3 Equitable total coloring of K_{p_1,p_2}, with $p_1 \neq p_2$

Here, we present equitable $(\Delta+1)$-total colorings for complete bipartite nonbalanced graphs by applying the same technique used in the previous section. It was established by Fu in [4] that $\chi''_e = \Delta + 1$ for K_{p_1,p_2} (with $p_1 \neq p_2$).

Let K_{p_1,p_2} with $p_1 \neq p_2$ be a bipartite complete nonbalanced graph. Assume, without loss of generality, that $p_1 < p_2$ and let $A = [a_{ij}]$ be a $(p_1+1) \times p_2$ matrix, in which the entry a_{ij} is the color assigned to the edge that has x_{1i} and x_{2j} as its ends if $1 \leq i \leq p_1$; and the color assigned to the vertex x_{2_j} if $i = p_1 + 1$. Similarly to the complete bipartite balanced case, to the element a_{ij} of A, assign the color $i+j-1 \mod (p_2)$ if $i+j-1 \not\equiv 0 \mod (p_2)$ and the color p_2, otherwise. To the vertices of X_1, assign the color p_2+1. It is easy to see that the proposed algorithm gives an equitable $(\Delta+1)$-total coloring of these graphs. Observe the following example when $p_1 = 3$ and $p_2 = 4$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

Theorem 2. *The algorithm gives an equitable $(\Delta + 1)$-total coloring of K_{p_1,p_2} $(p_1 \neq p_2)$.*

4 Equitable total coloring of $K_{r\times p}$ in which r is even (with $r \geq 4$) and p is odd

We begin this section by claiming that for these graphs, $\chi''_e \neq \Delta + 1$. Due to space restrictions, the proof is ommited. After, we provide an algorithm to show that $\chi''_e = \Delta + 2$, contributing to the ETCC.

Claim 1. *There exists no equitable total coloring with $\Delta + 1$ colors for $K_{r\times p}$ in which r is even (with $r \geq 4$) and p is odd.*

Claim 2 (Soifer, 2008 [5]). *Let K_n be the complete graph on n even ($n \geq 4$) vertices. Since this graph has a Δ-edge coloring, there exist $n-1$ disjoint perfect matchings in K_n. Let K_n be the complete graph on n odd ($n \geq 3$) vertices. Since this graph has a $(\Delta+1)$-edge coloring, there exist n disjoint matchings in K_n.*

The graph K_r is the complete graph having r vertices, in which r represents the number of parts of $K_{r\times p}$. We denote its matchings by R_i. Similarly, the graph K_p is the complete graph having p vertices, in which p represents the number of vertices in each part of $K_{r\times p}$. We denote its matchings by P_i.

Theorem 3. *The graph $K_{r\times p}$ with r even ($r \geq 4$) and p odd has $\chi''_e = \Delta + 2$.*

Proof. Using Claim 2 it is possible to organize all edge-coloring matrices as follows. If $R_i = \{v_a v_b, \cdots, v_c v_d\}$ is a perfect matching of K_r (there are $r-1$ disjoint perfect matchings in K_r), then let $S_i = \{A_{X_a X_b}, \cdots, A_{X_c X_d}\}$ be a set of edge-coloring matrices. Then, $|S_i| = |R_i|$.

To each set of matrices S_i, assign numbers from 1 to $p(r-1)$ as follows: the entries of the matrices in S_1 are the elements of the set $\{1, 2, 3, \cdots, p\}$; in S_2, elements of the set $\{p+1, p+2, \cdots, 2p\}$, and so on. In general, the matrices in S_i are elements of the set $\{(i-1)p+1, (i-1)p+2, \cdots, ip\}$.

Note that each one of the sets of edge-coloring matrices has p elements and that if they are taken modulo p, the sets become $\{1, 2, \cdots, p-1, 0\}$. Based on that, the distribution of elements in the edge-coloring matrices is done similarly to the distribution in the case of complete bipartite balanced graphs. Since $|S_i| = |R_i| = \frac{r}{2}$, each color from 1 to $p(r-1)$ was used $\frac{rp}{2}$ times in edges.

When we divide the matrices into $r-1$ groups, in each of those groups, part X_i appears precisely once. Therefore, when the same set of colors is assigned to one of the sets of edge-coloring matrices, no adjacent edges receive the same color.

We replace the secondary diagonal entries of $\frac{r-2}{2}$ matrices by the color $(r-1)p+1$ and $\frac{r-2}{2}$ matrices by the color $(r-1)p+2$, which have not been used yet. The replacement is done as follows: $A_{X_1X_2}, A_{X_2X_3}, \cdots, A_{X_{r-2}X_{r-1}}$ (note that these matrices are related to $S_1, S_2, \cdots, S_{r-2}$, respectively) have the entries in their secondary diagonals replaced alternately by $(r-1)p+1$ and $(r-2)p+2$. Note that by the algorithm to divide the matrices into sets, all the ones that have their entries changed are elements of $S_1, S_2, \cdots, S_{r-2}$, so their secondary diagonals received previously different colors.

Note that an element a_{ij} of $A_{X_kX_l}$ is in the secondary diagonal if it is such that $i+j=p+1$. Therefore, all entries in the secondary diagonal of one matrix receive the same color since $i+j-1=p+1-1=p$. In each group of edge-coloring matrices, the secondary diagonal will be $p, 2p, 3p, \cdots, (r-2)p$ by the coloring of edge-coloring matrices presented in the beginning of this proof.

The colors of the vertices will be the colors $(r-1)p+1$, $(r-1)p+2$ and the colors that have been changed in some of the secondary diagonals, which are $p, 2p, 3p, \cdots, (r-2)p$ since these colors must be represented in vertices that are ends of edges that are colored by $A_{X_1X_2}, A_{X_2X_3}, \cdots, A_{X_{r-2}X_{r-1}}$. To the vertices of X_i, for all $1 \leq i \leq r-2$, assign the color ip, to the vertices of X_{r-1}, assign the color $(r-1)p+1$ and to the vertices of X_r, assign the color $(r-1)p+2$.

The color $(r-1)p+1$ is used $\frac{(r-2)p}{2}$ times in edges and p times in the coloring of vertices, totalizing $\frac{rp}{2}$ times. The result is analogous for the color $(r-1)p+2$.

The colors that were originally in the secondary diagonal of some matrices and were replaced, were originally used p times in $\frac{r}{2}$ matrices. Then, they were replaced by a new color in p entries of a matrix. On the other side, those colors are used to color the p vertices of one of the parts of the partition of V. So, those colors are used $\frac{rp}{2}-p+p=\frac{rp}{2}$ in total. Each color is used precisely $\frac{rp}{2}$ times. Hence, the difference between the cardinalities of two color classes is 0 and this concludes the proof of the fact that the described algorithm gives an equitable $(\Delta+2)$-total coloring of $K_{r\times p}$ with r even ($r \geq 4$) and p odd. ■

5 Equitable total coloring of a part of graphs $K_{r\times p}$ in which r is odd and p is even

In this section, we provide an algorithm to show that $\chi''_e = \Delta + 1$ if $r \geq \frac{p}{2}$. Let $K_{r\times p}$ be a graph. We define *matching of distance* i as the set of edges $\{x_{ja}x_{(j+i),b}\}$ linking vertices of rows a and b ($1 \leq a, b \leq p, a \neq b$), for all $1 \leq j \leq r$ and $j+i$ is taken modulo r.

See an example of a matchings of distance 1 and 2 in the graph $K_{3\times 2}$ presented in Figure 1.

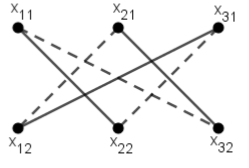

Figure 1: Matchings of distance 1 and 2 in the graph $K_{3\times 2}$

Since $j+i$ is taken modulo r, we have $j+i \leq r$, which implies that $i \leq r-1$ (since $j \geq 1$). Note that each matching of distance i has r edges. We define horizontal edges as the edges of the form $x_{ij}x_{kj}$, $1 \leq i, k \leq r$ and $1 \leq j \leq p$

Theorem 4. *The graph $K_{r\times p}$ with r odd and p even has $\chi''_e = \Delta + 1$ if $r \geq \frac{p}{2}$.*

Proof. If $p = 2$, assign the color i to the vertices $x_{i1} \in X_i$ and $x_{i2} \in X_i$, for all $i : 1, \cdots, r$. So, no adjacent vertices receive the same color. If a vertex x_{ij} received color i, take the matching R_l of K_r (Claim 2) in which v_i is the remaining vertex. Such matching has the form $R_l = \{v_av_b, \cdots, v_cv_d\}$. So, assign color i to the horizontal edges $x_{aj}x_{bj}, \cdots, x_{cj}x_{dj}$. Since v_i is not an end of any edge of R_l and the edges $x_{aj}x_{bj}, \cdots, x_{cj}x_{dj}$ are related to the matching R_l, this means that incident elements receive different colors.

Each color i that has been used in vertices was used twice (x_{i1} and x_{i2}). We have $|R_l| = \frac{r-1}{2}$. So, the color i has been used in $2\frac{r-1}{2}$ edges, totalizing $r+1$ elements.

Note that, so far, we have colored the horizontal edges and the vertices with $\frac{rp}{2}$ colors. Now, using different colors than the ones that have been already used, take $r-1$ colors to color matchings of distance $i = 1, \cdots, r-1$ of graph $K_{r\times 2}$. Since the cardinality of those matchings is r, each one of those $r-1$ colors are used r times and the difference between the cardinalities of two color classes is at most 1, as desired.

Now, consider the case $p \geq 4$ and $r \geq \frac{p}{2}$, in order to color the vertices of $K_{r\times p}$, since there are $p-1$ disjoint perfect matchings in K_p, in order to know how many times each matching P_i will be used to color the r parts of vertices of $K_{r\times p}$, divide r by $p-1$. By Euclidean division, there exist positive integers s and q such that $q(p-1)+s = r$. This means that we should use s matchings of K_p to color $q+1$ parts of the partition each and $p-1-s$ matchings to color q parts each.

The coloring of the vertices is done as follows: if $P_1 = \{v_a v_b, \cdots, v_c v_d\}$ and if we use it to color the vertices of X_1 then each of the following pairs of vertices receives a different color: x_{1a} and $x_{1b}, \cdots, x_{1c}$ and x_{1d}. The coloring of the horizontal edges is done similarly to the coloring of the horizontal edges in the case $K_{r\times p}$ with r and p being odd. After coloring the horizontal edges, we guarantee that if a color was used in a vertex of a certain row, than it was represented in all the vertices of that row. Now, this color needs to be represented in the vertices of the other rows.

To color the edges that are not horizontal, we begin with the colors that were used in vertices. If $P_1 = \{v_a v_b, v_c v_d, \cdots, v_g v_h\}$ and we used color i in the vertices x_{1a} and x_{1b} (admitting that P_1 was used to color the vertices in X_1), then we represent color i in the complement of P_1 with respect to $v_a v_b$, which we are going to denote by $P_1 - \{v_a v_b\} = \{v_c v_d, \cdots, v_g v_h\}$. Color i must be represented in the vertices of rows c and d, and so on, being represented in the rows determined by $\{v_cv_d, \cdots, v_gv_h\}$ above, until being used in the vertices of rows g and h. Color i is used in matchings of distance 1. We repeat the process to every color used in vertices always using the next available matching of distance. The other colors, that is, the ones not used in vertices, are assigned according to the available matchings of distances in each perfect matching P_i of K_p. The number of colors used only in edges of the graph is $\Delta + 1 - \dfrac{rp}{2}$. Indeed, since there are $(p-1)$ $P_i's$ and there are $r-1$ matchings of distance for each P_i, the number of matchings of distance is $(r-1)(p-1)$. From this number, we remove the matchings of distance colored with the colors of the vertices $[s(q+1)+(p-1-s)q]\left(\dfrac{p}{2}-1\right)$. As explained above, s matchings are used in the coloring of $q+1$ parts and $p-1-s$ matchings are used in the coloring of q parts. It is easy to check that each time we use a matching P_i to color a part, we use $\dfrac{p}{2}-1$ matchings of distance when applying a color of a pair of vertices to the complement of P_i: $(r-1)(p-1)-[s(q+1)+(p-1-s)q]\left(\dfrac{p}{2}-1\right) = rp-r-p+1+(-sq-s-pq+q+sq)\left(\dfrac{p}{2}-1\right) = rp-r-p+1-(q(p-1)+s)\left(\dfrac{p}{2}-1\right) = rp-r-p+1-r\left(\dfrac{p}{2}-1\right) = rp-r-p+1-\dfrac{rp}{2}+r = \dfrac{rp}{2}-p+1 = \Delta+1-\dfrac{rp}{2}$.

Note that there are $r-1$ matchings of distance j and $|P_i| = \frac{p}{2}$. So, the complement of P_i has size $\frac{p}{2}-1$. In order to have available matchings of distance to complete the coloring, we must have $\frac{p}{2}-1 \leq r-1$, which implies that $r \geq \frac{p}{2}$.

By construction we get that no adjacent or incident elements receive the same color. Also, we have that $\frac{rp}{2}$ colors are used in 2 vertices and then, represented in the other vertices through edges that have them as ends. Since each edge has 2 ends, we get that those colors are used in $\frac{rp-2}{2}$ edges, totalizing $\frac{rp}{2}+1$ elements. The other $\frac{rp}{2}-p+1$ colors are used in $\left\{\frac{rp^2(r-1)}{2}-\left(\frac{rp-2}{2}\right)\frac{rp}{2}\right\} \div \frac{rp}{2} = \frac{rp}{2}-p+1$ perfect matchings of the graph $K_{r\times p}$, that is, in $\frac{rp}{2}$ elements. We conclude that the difference between two different color classes is at most 1, as desired. ■

Observe the equitable $(\Delta+1)$-total coloring of $K_{3\times 4}$ exhibited below.

We have matchings $P_1 = \{v_1v_2, v_3v_4\}, P_2 = \{v_2v_3, v_1v_4\}$ and $P_3 = \{v_3v_4, v_1v_2\}$ of $K_{p=4}$. The coloring of the vertices of X_1 is related to P_1, the vertices of X_2 to P_2, and the vertices of X_3 to P_3. Observe the coloring of vertices and horizontal edges in Figure 2.

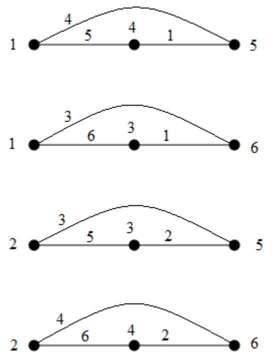

Figure 2: Coloring of vertices and horizontal edges of $K_{3\times 4}$

Colors 1 to 6 are also used in the following edges: color 1 in the matching of distance 1 linking vertices of rows 3 and 4; color 2 in the matching of distance 1 linking vertices of rows 1 and 2; color 3 in the matching of distance 1 linking vertices of rows 1 and 4; color 4 in the matching of distance 1 linking vertices of rows 2 and 3; color 5 in the matching of distance 1 linking vertices of rows 2 and 4; color 6 in the matching of distance 1 linking vertices of rows 1 and 3. Colors 7, 8 and 9 are used in the following edges: $7 \to P_1$ dist 2, $8 \to P_2$ dist 2 and $9 \to P_3$ dist 2.

References

[1] M. Behzad. Graphs and their chromatic numbers. PhD Thesis. Michigan State University, 1965.

[2] J. C. Bermond. Nombre chromatique total du graphe r-parti complet, J. London Math. Soc. 9, 1974, 279–285.

[3] S. Dantas and C. M. H. de Figueiredo and G. Mazzuoccolo and M. Preissmann and V. F. dos Santos and D. Sasaki. On the equitable total chromatic number of cubic graphs, Discrete Appl. Math. 209, 2016, 84–91.

[4] H. L. Fu. Some results on equalized total coloring. Congr. Numer. 102, 1994, 111–119.

[5] A. Soifer. The Mathematical Coloring Book. Springer-Verlag. 2008.

[6] V. G. Vizing. Some unsolved problems in graph theory. Russian Math. Surveys 23 (6), 1968, 125–141.

[7] W. F. Wang. Equitable total colouring of graphs with maximum degree 3, Graphs Combin. 18, 2002, 677-685.

IME, Universidade Federal Fluminense, Brazil
andersongs@hotmail.com.br
sdantas@im.uff.br

IME, Universidade do Estado do Rio de Janeiro, Brazil
diana.sasaki@ime.uerj.br