

On the In-Neighbor Convexity

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Abstract

In this work, we study the in-neighbor convexity in the directed graph class. We work with the following parameters: in-neighbor convexity number, in-neighbor number and in-neighbor hull number. We present results concerning directed acyclic graphs, directed cyclic graphs and transitive graphs.

1 Introduction

The concept of graph convexity has been studied in many works and conceptions [1, 2, 3, 5]. This work propose a new convexity called in-neighbor convexity, such convexity can be applied in the process of spread of influence through a social network modeled by graphs. This process has been studied in many fields such as epidemiology [4], sociology, economics and computer science [6].

Let $D(V, A)$ be a simple directed graph, the indegree of a vertex v is denoted by $d^-(v)$ and the outdegree of v is denoted by $d^+(v)$. The set of direct predecessors of v in D is denoted by $N^-(v)$. A vertex with $d^-(v) = 0$ is a source. A directed graph is acyclic if it has no directed

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cycles. Every directed acyclic graph has at least one source. Let $f : V(D) \rightarrow \{0, 1, 2, \dots, j\}$ be a function. A set $S \subset V(D)$ is in-neighbor convex if $\forall v \notin S$ has $|N^-(v) \cap S| < f(v)$. The in-neighbor convexity number of D , $c_p(D)$, is the cardinality of the largest proper in-neighbor convex set of D . The in-neighbor interval of a set $S \subset V(D)$, $I_p(S) = S \cup \{v \in V(D) : |N^-(v) \cap S| \geq f(v)\}$. When $I_p(S) = V(D)$, S is called in-neighbor set. The in-neighbor number of D , $p_p(D)$, is the cardinality of the smallest in-neighbor set. Let S be a subset of $V(D)$, $I_p^+(S) = S_i$ where $S_1 = I_p(S)$, $S_2 = I_p(S_1)$, $S_3 = I_p(S_2)$, ..., $S_i = I_p(S_{i-1})$ and S_i is an in-neighbor convex set. The set S is called in-neighbor convex hull. When $I_p^+(S) = V(D)$, then S is an in-neighbor hull of D . The in-neighbor hull number, $h_p(D)$, is the cardinality of the smallest in-neighbor hull of D .

Assume that a vertex v is contaminated if v is in the subset S or $I_p^+(S)$. A vertex becomes contaminated if it has at least $f(v)$ contaminated in-neighbors.

In this work we studied the In-Neighbor Convexity for directed acyclic graphs and directed cyclic graphs.

2 Results

Theorem 1. Let $D(V, A)$ be a directed acyclic graph, then:

$$c_p(D) = \begin{cases} |V(D)| - 1, & \text{if } \exists v : |N^-(v)| < f(v); \\ 0, & \text{if } \forall v : |N^-(v)| \geq f(v). \end{cases}$$

Proof. In the first case ($|V(D)| - 1$), in order to find the proper convex in-neighbor set of D is sufficient to exclude one vertex that has $|N^-(v)| < f(v)$. The vertex v will never have enough neighbors in S to be contaminated. Therefore $c_p(D) = |V(D)| - 1$.

In the second case, we show that $I_p^+(\emptyset) = V(D)$, thus $c_p(D) = 0$. First, we prove that $I_p^+(\emptyset) \neq \emptyset$. Since D is an acyclic directed graph, D has a source, called v . By the property $|N^-(v)| \geq f(v)$, it follows that $f(v) = 0$ for all the sources. Observe that $\{x \in V(D) : f(x) = 0\} \subseteq I_p^+(\emptyset)$. Thus,

$I_p^+(\emptyset)$ has all the sources of D . Suppose that $I_p^+(\emptyset) \neq V(D)$, therefore exists at least a vertex v of D that $v \notin I_p^+(\emptyset)$, then v is not a source. By the property $|N^-(v)| \geq f(v)$, we can conclude that $|N^-(v)| + 1 - f(v)$ direct predecessors of v do not belong to $I_p^+(\emptyset)$. The same can be concluded about these vertices and so on. Since the graph is acyclic, eventually we will find the beginning of the path, a source, that does not belong to $I_p^+(\emptyset)$. Contradiction. ■

The in-neighbor number problem is NP-Complete for directed acyclic graphs. This problem can be reduced by applying the Propositional Satisfiability Problem, also known as SAT.

Consider a boolean expression written only by using the logical operators AND (\wedge), OR (\vee), NOT (\neg), variables (x_1, x_2, \dots, x_i) and parenthesis. A literal corresponds to a single variable or its complement; a clause with the set of literals grouped by the disjunction symbol (OR). Therefore, the formulas will be a conjunction (AND) of clauses.

The following shows the reduction of it:

Problem 1. IN-NEIGHBOR SET

INSTANCE: Graph $D = (V(D), A(D))$, positive integer $k \leq n$

QUESTION: Is there an in-neighbor set of size k or less in D , i.e., is there a subset $V' \subseteq V(D)$, such that $|V'| \leq k$ and every $u \in V(D) - V'$ has at least $f(u)$ predecessors in V' ?

Theorem 2. The IN-NEIGHBOR SET problem is NP-complete when considering the directed acyclic graphs.

Proof. An in-neighbor set lies in NP, because considering a given certificate (a subset S of vertices of D), it is possible to answer in a polynomial time if S has the appropriated size and if all vertices that do not belong to S have $f(x)$ predecessors in S . In order to complete the proof, we describe a reduction from SAT. Let $\mathcal{C} = C_1 \wedge C_2 \wedge \dots \wedge C_i$ be an arbitrary instance of SAT. We construct a graph D in the following way: for each clause C_j of \mathcal{C} creates a vertex c_j in $V(D)$. For each variable \mathcal{C} create a set of

vertices $\{x_l, \bar{x}_l, v_l, u_l\}$ and the edges $\{(x_l u_l), (\bar{x}_l u_l), (v_l x_l), (v_l \bar{x}_l)\}$. Assign to all vertices $f(x) = 1$. Finally, if a variable x_l/\bar{x}_l appears in the clause C_i , add the edge $(x_l c_i)/(\bar{x}_l c_i)$ in D .

First, we assume that \mathcal{C} is satisfiable, and consider a minimum satisfying truth assignment S . Suppose S is a minimum set of true variables that satisfy \mathcal{C} . Let P be the in-neighbor set of D and V the set of all vertices v of D . So $P = S \cup V \cup \{u_l \in D \mid x_l, \bar{x}_l \notin S\}$. Since all clauses of \mathcal{C} have, at least, one true variable, then we have that each c_i has at least a predecessor in P . The set of V ensures the predecessor of x and \bar{x} . The set S provides predecessors for some vertex u , those that do not have predecessor in S belongs to P . So $k = 2i$, which means, i vertices v , k vertices of S and $i - k$ vertices of u .

For the converse, we assume that S is an in-neighbor set of D . Then S has all vertices v of D which ensures one predecessor for each x and \bar{x} , $|S| = i$. To be able to have an in-neighbor set it is required to put in S the vertices c or one of its predecessors. Since we want a minimum in-neighbor set, we will put in S one predecessor of c_i , because this predecessor is also a predecessor of u_i . There can also exist vertices u without predecessors, which for the matter of making it clear, we will add then to S . So $|S| = (i) + (k) + (i - k) = 2i$. Then the truth set of \mathcal{C} is $\{x/\bar{x} \mid x/\bar{x} \in S\}$. ■

Theorem 3. Let D be a directed acyclic graph then $h_p(D) = |S|$, where S is the set of all vertices v of D such that $|N^-(v)| < f(v)$.

Proof. Let $D(V, A)$ be a directed acyclic graph and S the set of all vertices of $D : |N^-(v)| < f(v)$. Suppose that S is not a in-neighbor hull of D . Then there is a vertex of D , $v : v \notin I_p^+(S)$. Therefore v has $|N^-(v)| + 1 - f(v)$ direct predecessors that do not belong to $I_p^+(S)$. These vertices clearly have $|N^-(x)| \geq f(x)$ and the same applies to them, and thus iteratively. Since the graph is acyclic, we will reach to the beginning of a path, a source, which or belongs to S or to $I_p^+(S)$. Contradiction.

Clearly, S is minimum. ■

Next, we consider directed cyclic graphs. We show algorithms that solve the problem for the in-neighbor convexity number, in-neighbor number and in-neighbor hull number.

Algorithm 1

Input: A directed cyclic transitive graph $D = (V, A)$ such that $\forall v \in V(D) : f(v) = k$, and k is a constant integer greater than 1.

Output: $c_p(D)$.

1. Find the strongly connected components C_i of the directed graph D .
2. Reduce each strongly connected component C_i to a single vertex v_i and assign a weight P_i to each one, where $P_i := |V(C_i)|$.
3. For each v_i create the oriented edge $v_i v_j$, where $v_i \in V(C_i)$ and $v_j \in V(C_j)$, if there is at least one vertex of C_i that is the direct predecessor of some vertex of C_j in the original directed graph.
4. Choose a v_i source with the lowest P_i and do $m := P_i$.
5. Do $c_p(D) := |V(D)| - m + f(v) - 1$.

Lemma 1. Let D be a directed, strongly connected and transitive graph, then D is a complete graph.

Proof. Suppose that D is not complete. By definition, a directed graph is strongly connected if there exists, for every pair u, v of D , a directed path from u to v and vice versa. Since D is not complete there is at least one edge uv that does not belong to $A(D)$. However D is transitive and by definition if there is a path between the vertices u and v then the edge uv belongs to $A(D)$, therefore, there is an edge between each pair of vertices of D , contradiction. D is complete. ■

When a strongly connected component C_i is reduced to a source vertex v_i in the Algorithm 1 let's call it a source component.

Lemma 2. Let D be a directed cyclic transitive graph, C_i a strongly connected source component of D , C_j a strongly connected non-source

component of D , C_i is predecessor of C_j ; there is the edge uv between each pair of vertices u and v , where $u \in V(C_i)$ and $v \in V(C_j)$.

Proof. Suppose there is no edge uv in $A(D)$, where $u \in V(C_i)$ and $v \in V(C_j)$. According to Lemma 1 the strongly connected components of D are complete graphs, and as C_i is predecessor of C_j there is a path from u to v . Since D is transitive, by definition, the edge uv belongs to $A(D)$ if there is a path between the vertices u and v , contradiction. ■

As a consequence of Lemma 2 is easy to see that if D is directed cyclic transitive graph, every non-source component has a source component as direct predecessor.

Theorem 4. The number $c_p(D)$ found by the Algorithm 1 is the cardinality of the largest proper in-neighbor convex set of D , where D is a directed cyclic transitive graph such that $\forall v \in V(D) : f(v) = k$, and k is a constant integer greater than 1.

Proof. Let C_i be a transitive strongly connected component found by the Algorithm 1. According the Lemma 1, C_i is a complete directed graph, then we can consider only $f(v) - 1$ vertices of C_i , otherwise, the component will become completely contaminated.

Suppose the $c_p(D)$ found by the Algorithm 1 is not the cardinality of the largest proper in-neighbor set S of D , then there is a proper in-neighbor set S' of D where $|S'| > |S|$. Therefore, there is C'_i such that $|C'_i| < |C_i|$, where C_i is the source component with the lowest cardinality of D . We know that C'_i is not a source, thus, according the Lemma 2, C'_i will be contaminated by its predecessor, then S' is not a proper in-neighbor set of D , contradiction. ■

Algorithm 2

Input: A directed cyclic transitive graph $D = (V, A)$ where $\forall v \in V(D) : f(v) = k$, and k is a constant integer greater than 1.

Output: $p_p(D)$.

1. Find the strongly connected components C_i of the directed graph D .
2. Reduce each strongly connected component C_i to a single vertex v_i and assign a weight P_i to each one, where $P_i := |V(C_i)|$.
3. For each v_i create the oriented edge $v_i v_j$, where $v_i \in V(C_i)$ and $v_j \in V(C_j)$, if there is at least one vertex of C_i that is the direct predecessor of some vertex of C_j in the original directed graph.
4. Count the amount of v_i sources that exist in the graph and do q receive that amount.
5. Do $p_p(D) := q * f(v)$.

Theorem 5. The in-neighbor number $p_p(D)$ found by the Algorithm 2 is the cardinality of the smallest in-neighbor set of D , where D is a directed, cyclic and transitive graph such that $\forall v \in V(D) : f(v) = k$, and k is a constant integer greater than 1.

Proof. According the Lemma 1, the strongly connect components of the graph are complete graphs, then it suffices to contaminate $f(v)$ vertices of each source component to contaminate the whole component and, according the Lemma 2, these vertices will also contaminate the non-source components of the graph. ■

Theorem 6. Let D be a directed cyclic transitive graph where $\forall v \in V(D) : f(v) = k$, and k is a constant integer greater than 1, then $h_p(D) = p_p(D)$.

Proof. In order to get the graph contaminated we must contaminate $f(v)$ vertices of each strongly connected source component, these vertices will contaminate the source and non-source components of the graph, as shown in Theorem 5. Therefore $h_p(D) = p_p(D)$. ■

Algorithm 3

Input: A directed cyclic graph $D = (V, A)$ where $\forall v \in V(D) : f(v) = 1$.

Output: $c_p(D)$.

1. Find the strongly connected components C_i of the directed graph D .
2. Reduce each strongly connected component C_i to a single vertex v_i and assign a weight P_i to each one, where $P_i := |V(C_i)|$.
3. For each v_i create the oriented edge $v_i v_j$, where $v_i \in V(C_i)$ and $v_j \in V(C_j)$, if there is at least one vertex of C_i that is the direct predecessor of some vertex of C_j in the original graph.
4. Choose a v_i source with the lowest P_i and do $m := |P_i|$.
5. Do $c_p(D) := |V(D)| - m$.

Theorem 7. The number $c_p(D)$ found by the Algorithm 3 is the cardinality of the largest proper in-neighbor convex set of D , where D is a directed cyclic graph such that $\forall v \in V(D) : f(v) = 1$.

Proof. After the graph reduction performed in the Algorithm 3, it becomes an directed acyclic graph. According the Theorem 1, it suffices to remove a single vertex that has $|N^-(v)| < f(v)$, in this case, a source component. In order for $c_p(D)$ to be maximum, it is necessary to choose the lowest weight source component. Since this component is strongly connected and $\forall v \in V(D) : f(v) = 1$, clearly, none of its vertices are in S the largest proper in-neighbor convex set that satisfies $c_p(D) = |S|$. ■

Algorithm 4

Input: A directed cyclic graph $D = (V, D)$ where $\forall v \in V(D) : f(v) = 1$.

Output: $h_p(D)$.

1. Find the strongly connected components C_i of the directed graph D .
2. Reduce each strongly connected component C_i to a single vertex v_i and assign a weight P_i to each one, where $P_i := |V(C_i)|$.
3. For each v_i create the oriented edge $v_i v_j$, where $v_i \in V(C_i)$ and $v_j \in V(C_j)$, if there is at least one vertex of C_i that is the direct predecessor of some vertex of C_j in the original directed graph.

4. Count the number of sources that exist in the directed graph and do q receive that amount.
5. Do $h_p(D) := q$.

Theorem 8. The in-neighbor hull number $h_p(D)$ found by the Algorithm 4 is the cardinality of the smallest in-neighbor hull of D , where D is a directed cyclic graph such that $\forall v \in V(D) : f(v) = 1$.

Proof. Suppose the $h_p(D)$ found by the Algorithm 4 is not the smallest cardinality of in-neighbor hull of D , then there is a $h'_p(D) < h_p(D)$, where $h'_p(D) = |S'|$ and $h_p(D) = |S|$, such that S' and S are in-neighbor hull sets of D . According the Algorithm 4, each C_i source has one vertex in S . Since S' must have one vertex less than S , there is a C_i source that does not have vertices in S' . Since this C_i is a source, none of its vertices has predecessors in S' and will never be contaminated, therefore S' is not a in-neighbor hull, contradiction. ■

3 Conclusions

In this work, we show for the class of directed acyclic graphs intervals for in-neighbor convexity number and in-neighbor hull, it is also shown for this class that the in-neighbor number is NP-Complete. Moreover, for directed cyclic graphs we have created algorithms that answer the three parameters of In-Neighbor Convexity for the class of directed transitive graphs where $\forall v \in V(D) : f(v) = k$, and k is a constant integer greater than 1, as well as algorithms that solve the in-neighbor convexity number and in-neighbor hull number for directed cyclic graphs that have $f(v) = 1$.

References

- [1] Centeno, C. C., Dourado, M. C., Szwarcfiter, J. L. *On the convexity of paths of length two in undirected graphs*. Electronic Notes in Discrete Mathematics, v. 32, 2009, p. 11-18.

- [2] Changat, M., Mulder, H. M., Sierksma, G. *Convexities related to path properties on graphs*. Discrete Mathematics, v. 290, n. 2, 2005, p. 117-131.
- [3] Chartrand, G., Fink, J. F., Zhang, P. *Convexity in oriented graphs*. Discrete applied mathematics, v. 116, n. 1, 2002, p. 115-126.
- [4] Dezső, Z., Barabási, A. L. *Halting viruses in scale-free networks*. Physical Review E, v. 65, n. 5, 2002, p. 055103.
- [5] Dourado, M. C., Protti, F., Szwarcfiter, J. L. *Algorithmic aspects of monophonic convexity*. Electronic Notes in Discrete Mathematics, v. 30, 2008, p. 177-182.
- [6] Kempe, D., Kleinberg, J. M., Tardos, É. *Maximizing the Spread of Influence through a Social Network*. Theory of Computing, v. 11, n. 4, 2015, p. 105-147.

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