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Constructing pairs of Laplacian equienergetic threshold graphs

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Abstract

The Laplacian energy of a graph G on n vertices and m edges is defined as the sum of absolute values of the differences between each Laplacian eigenvalue of G and the average degree $2m/n$. In this work we construct pairs of threshold graphs of same order, with same Laplacian energy and different sets of Laplacian eigenvalues.

1 Introduction

Let G be a simple and undirect graph on vertices $v_1, \ldots, v_n$. The *Laplacian matrix* $L = L(G)$ of G is the $n \times n$ matrix for which the entry L_{ii} is the degree of vertex v_i, $1 \leq i \leq n$, and the entries L_{ij} are -1, if vertex v_i and v_j are adjacent in G, and 0 otherwise. The matrix L is symmetric, positive-semidefinite and always has 0 as an eigenvalue. For these and other properties of the L matrix, see [Mer94]. As usual, we denote the L-eigenvalues as $\mu_1, \mu_2, \ldots, \mu_n$, where $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n = 0$; these real numbers constitute the *Laplacian spectrum* of G. Two graphs are

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cospectral if they share the same Laplacian spectrum, otherwise they are *non-cospectral.*

If G has m edges, then $2m/n$ is its *average degree* and the *Laplacian energy* of G is defined as $LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$. After introduced by Gutman and Zhou in [GZ06], this concept has been extensively investigated. In particular, the problem of constructing families of non-cospectral graphs with same order and equal Laplacian energy (called *Laplacian-equienergetic graphs* or simply *L-equienergetic graphs*) has already been studied in [Ste09] and [FHT14], but a general characterization remains an open problem.

Our investigation concerns the construction of L-equienergetic threshold graphs.

Threshold graphs were introduced by Chvátal and Hammer [CH77] and independently, by Henderson and Zalcstein [HZ77], in 1977. They constitute an important class of graphs due to their numerous applications in diverse areas. Threshold graphs can be characterized in many ways. In this paper, a *threshold graph* is obtained through an iterative process which starts with an isolated vertex, and where, at each step, either a new isolated vertex is added, or a vertex adjacent to all previous vertices (*dominating vertex*) is added. Following this construction, a threshold graph can be represented by a string of $0's$ and $1's$, corresponding respectively to isolated vertices and *dominating* vertices. The threshold graph constructed in Figure 1 has 0-1-string $(0, 0, 0, 1, 0, 0, 1)$.

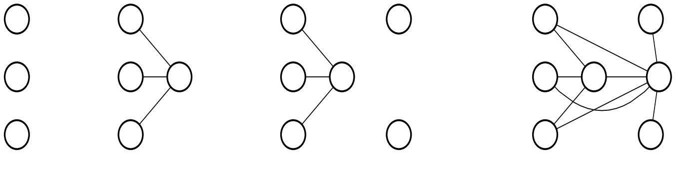

Figure 1: Constructing the threshold graph with string $(0, 0, 0, 1, 0, 0, 1)$.

The number of characters 1 in the string (that is, the number of dominating vertices in the graph) is called the *trace* of the graph and denoted by $T = T(G)$. The dominating vertices with the first vertex inserted in the graph constitute a clique. Thus, the clique number of the graph is equal to $T + 1$. For example, the graph in Figure 1 has clique number 3 $(= T + 1)$. The other 4 $(= n - T - 1)$ vertices constitute an independent set.

The 0-1-string of a threshold graph also provides its degree sequence $d = [d_1, d_2, \ldots, d_i, \ldots, d_n]$, where $d_1 \geq d_2 \geq \ldots \geq d_n$. For example, the graph in Figure 1 has degree sequence $[6, 4, 2, 2, 2, 1, 1]$.

The following known result furnishes the sequence of L-eigenvalues of a threshold graph from its degree sequence, showing that all of them are integers.

Theorem 1.1 ([Mer94])**.** Let G be a threshold graph on n vertices, with trace T and degree sequence $[d_1, d_2, \ldots, d_n]$ arranged in non increasing order. Then for the Laplacian eigenvalues of G it holds that $\mu_i = d_i + 1$, if $1 \leq i \leq T$, $\mu_i = d_{i+1}$, if $T + 1 \leq i \leq n - 1$, and $\mu_n = 0$.

In [VDVJT13], the authors establish an explicit formula to compute the Laplacian energy of threshold graphs satisfying certain hypothesis.

Theorem 1.2 ([VDVJT13])**.** Let G be a threshold graph on n vertices, m edges and trace T with $3 \leq T \leq n - 1$. If the Laplacian eigenvalues of G satisfy $\mu_T \geq 2m/n \geq \mu_{T+1}$ then the Laplacian energy of G is given by

$$LE(G) = T^2 + T + \left(1 - \frac{2T}{n}\right) 2m\,.$$

2 Families of threshold graphs satisfying conditions of Theorem 1.2

Given integers $n \geq 3$ and $T \geq 1$, the graph on n vertices obtained by attaching $n - T - 1$ pendent vertices to the same vertex of the complete graph K_{T+1} is said to be the *pineapple on n vertices and trace T* and denoted $P_{n,T}$. Its 0-1 string is $(0, \underbrace{1,1,\ldots,1}_{T-1}, \underbrace{0,0,\ldots,0}_{n-T-1}, 1)$. For fixed n and T, it is the threshold graph with least number of edges. The graph $P_{n,T}$ has degree sequence $[n-1, T\ (T \text{ times}), 1((n-T-1) \text{ times})]$, $\frac{1}{2}(T^2+T)+(n-T-1)$ edges and its Laplacian spectrum is n, $T+1$ $((T-1)$ times), 1 $((n-T-1)$ times) and 0.

In what follows, n and T are integers such that $4 \leq T \leq n-2$.

2.1 A known family

In [VDVJT13], the authors exhibit a family of threshold graphs satisfying the hypothesis of Theorem 1.2. Denote by $\mathcal{G}_{n,T}$ the family of threshold graphs G_t on n vertices and trace T, where $G_0 = P_{n,T}$ and for t, $1 \leq t \leq (T-1)(n-T-1)$, G_t is obtained from G_0 by the addition of t edges to specific vertices of G_0 in such way that the 0-1-strings of the graphs obtained are:

$$s_0 = (0, \underbrace{1,\cdots,1}_{T-1}, \underbrace{0,0,\cdots,0}_{n-T-1}, 1); \quad s_1 = (0, \underbrace{1,\cdots,1}_{T-2}, 0,1,0,0,\cdots,0,1);$$

$$s_2 = (0, \underbrace{1,\cdots,1}_{T-2}, 0,0,1,0,0,\cdots,0,1); \quad \cdots \quad s_{(n-T-1)} = (0, \underbrace{1,\cdots,1}_{T-2}, 0,0,\cdots,0,1,1);$$

$$s_{(n-T-1)+1} = (0, \underbrace{1,\cdots,1}_{T-3}, 0,1,0,0,\cdots,0,1,1); \quad \cdots$$

$$s_{2(n-T-1)} = (0, \underbrace{1,\cdots,1}_{T-3}, 0,\cdots 0,1,1,1); \quad \cdots \quad s_{(T-2)(n-T-1)} = (0,1,0,\cdots,0 \underbrace{1,\cdots,1}_{T-2}, 1);$$

$$s_{(T-2)(n-T-1)+1} = (0,0,1,0,\cdots,0, \underbrace{1,\cdots,1}_{T-2}, 1); \quad \cdots$$

$$\cdots s_{(T-1)(n-T-1)} = (0,\cdots,0, \underbrace{1,\cdots,1}_{T-1}, 1).$$

In [VDVJT13], conditions on the number of inserted edges t are given in order to identify the graphs G_t of $\mathcal{G}_{\backslash,\mathcal{T}}$ satisfying the hypothesis of Theorem 1.2. The introduction of the parameters $t^\flat$ and $t^\sharp$, both depending on n and T, provides bounds on t in order to compute the Laplacian energy of these graphs.

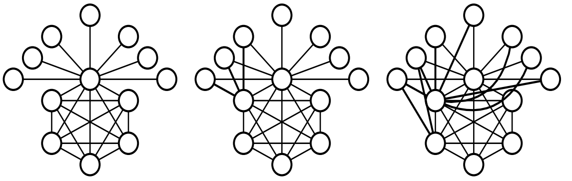

Figure 2: The graphs $G_0 = P_{13,5}$, G_3 and G_9 of family $\mathcal{G}_{13,5}$.

Theorem 2.1 ([VDVJT13])**.** For fixed n and T, let $t^\flat = \frac{1}{2}(n-T)(T-1)+1$ and $t^\sharp = \frac{1}{n-2}\left[(1-n)T^2 + (n^2-1)T + (3-2n)n - 2\right]$. Thus $1 \le t^\flat \le (n-T-1)(T-2) \le t^\sharp \le (n-T-1)(T-1)$. Furthermore, for each integer t, $1 \le t \le (n-T-1)(T-1)$, if $t \le t^\flat$ or $t \ge t^\sharp$ then $LE(G_t) = T^2 + T + \left(2 - \dfrac{4T}{n}\right) m_t$, where m_t denotes the number of edges of the graph G_t of $\mathcal{G}_{\backslash,\sqcup}$.

As pointed out in [VDVJT13], at least half of the threshold graphs in $\mathcal{G}_{\backslash,\mathcal{T}}$ satisfy the conditions of Theorem 2.1.

2.2 A new family

In this work, we present the construction of another family of threshold graphs for which Theorem 1.2 still holds.

The graphs F_t of the family $\mathcal{F}_{n,T}$ have n vertices and trace T. They are also obtained from the pineapple $F_0 = P_{n,T}$ by the insertion of t edges ($1 \le t \le (T-1)(n-T-1)$) in such way theirs respective 0-1-strings $s_1, \dots, s_t \dots s_{(n-T-1)(T-1)}$ are:

$$s_1 = (0, \underbrace{1, \cdots, 1}_{T-2}, 0, 1, 0, \cdots, 0, 1);\; s_2 = (0, \underbrace{1, \cdots, 1}_{T-3}, 0, 1, 1, 0, \cdots, 0, 1);$$

$$s_3 = (0, \underbrace{1, \cdots, 1}_{T-4}, 0, 1, 1, 1, 0, \cdots, 0, 1);\; \cdots\; s_{(T-1)} = (0, 0, \underbrace{1, \cdots, 1}_{T-1}, 0, \cdots, 0, 1);$$

$$s_{(T-1)+1} = (0, 0, \underbrace{1, \cdots, 1}_{T-2}, 0, 1, 0, \cdots, 0, 1);\; s_{(T-1)+2} = (0, 0, \underbrace{1, \cdots, 1}_{T-3}, 0, 1, 1, 0, \cdots, 0, 1);$$

$$\cdots\; s_{2(T-1)} = (0, 0, 0, \underbrace{1, \cdots, 1}_{T-1}, 0, \cdots, 0, 1);\; s_{2(T-1)+1} = (0, 0, 0, \underbrace{1, \cdots, 1}_{T-2}, 0, 1, 0, \cdots, 0, 1);$$

$$\cdots\; s_{(n-T-1)(T-1)-1} = (0, \cdots, 0, 1, 0, \underbrace{1, \cdots, 1}_{T-1}) \cdots$$

$$\cdots \ldots s_{(n-T-1)(T-1)} = (0, \cdots, 0, 1, 1, 1, \cdots, 1, 1)$$

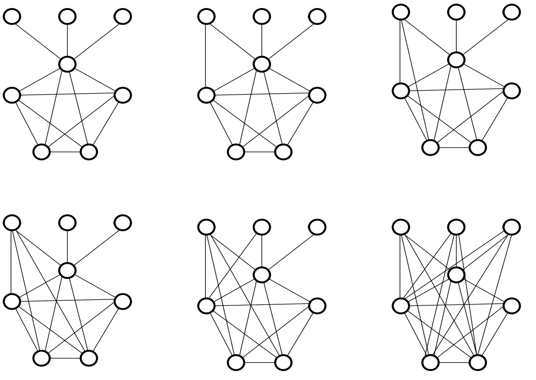

Figure 3: Graphs in family $\mathcal{F}_{8,4}$: $F_0 = P_{8,4}$, F_1, F_2, F_3, F_4 and F_9.

Reasoning analogously to Proposition 2 of [VDVJT13] we can prove that:

Theorem 2.2. For fixed n and T, let $t_1 = \frac{T(T-1)}{n-2} + 1$ and $t_2 = \frac{1}{2}(n-T)(T-1) + 1 - \frac{n}{2}$. Thus $1 \leq t_1 \leq (n-T-1)(T-2) \leq t_2 \leq (n-T-1)(n-1)$. Furthermore, for all t, $1 \leq t \leq (n-T-1)(T-1)$, if $1 \leq t \leq t_1$ or $t \geq t_2$ then $LE(F_t) = T^2 + T + \left(2 - \dfrac{4T}{n}\right) m_t$, where m_t denotes the number of edges of the graph F_t.

Proof: Analogously to the proof of Theorem 2.1 (Proposition 2 of [VDVJT13]), the three statements below must be proved:

1. $\mu_T \geq \frac{2m_t}{n}; \forall 1 \leq t \leq (n-T-1)(T-1)$
2. for $1 \leq t \leq (T-2)$, $\mu_{T+1} \leq \frac{2m_t}{n}$ if and only if $t \leq t_1$;
3. for $(T-1) \leq t \leq (n-T-1)(T-1)$, $\mu_{T+1} \leq \frac{2m_t}{n}$ if and only if $t \geq t_2$.

■

Numerical experiments show that at least half of the threshold graphs in $\mathcal{F}_{\backslash,\mathcal{T}}$ satisfy the conditions of Theorem 2.2.

3 Constructing pairs of L-equienergetic threshold graphs

Threshold graphs are determined by their L-spectra, that is, if G and H are two L-cospectral threshold graphs then they are isomorphic [Mer94].

In [Ste09], the author constructs large sets of L-equienergetic threshold graphs all of them having equal trace. Pairs of L-equienergetic threshold graphs with different traces are exhibited in [VDVJT13]. In what follows, we present infinite pairs of L-equienergetic threshold graphs.

In the next result, we present pairs of L-equienergetic graphs where both graphs were constructed by the same way although with different traces.

Theorem 3.1. For fixed n and T, let $T' = n - T$. For t such that $1 \leq t \leq (n-T-1)(T-1)$, let $s = (n-T'-1)(T'-1)-(t-1)$. If $t \leq t_1(T)$ of Theorem 2.2 then $LE(F_t) = LE(F_s)$, where F_t belongs to family $\mathcal{F}_{\backslash,\mathcal{T}}$ and F_s to family $\mathcal{F}_{\backslash,\mathcal{T}'}$.

Proof: Firstly, we note that F_t and F_s have different spectra since theirs traces are distinct. By algebraic manipulations it can be verified that if $t \leq t_1(T)$ then $s \geq t_2(T')$ and so, $LE(F_t)$ and $LE(F_s)$ can be obtained by Theorem 2.2. Let m_t and m_s denote the number of edges of F_t and of F_s, respectively. Then

$$LE\left(F_t\right) = T^2 + T + \left(1 - \frac{2T}{n}\right) 2m_t =$$

$$= T^2 + T + \left(1 - \frac{2T}{n}\right) 2\left(\frac{T\left(T+1\right)}{2} + n - T - 1 + t\right) =$$

$$= 2\left[T^2 - 2T + n - \frac{T^2}{n}\left(T-1\right) + \left(t-1\right)\left(1 - \frac{2T}{n}\right)\right].$$

Since

$$2m_s = 2\left(\frac{T^{'}\left(T^{'}+1\right)}{2} + n - T^{'} - 1 + \left(T^{'}-1\right)\left(n - T^{'} - 1\right) - \left(t-1\right)\right) =$$

$$= n^2 - T^2 + T - n + 2 - 2t$$

then $$\left(1 - \frac{2T^{'}}{n}\right) 2m_s = \left(\frac{2T}{n} - 1\right)\left(n^2 - T^2 + T - n + 2 - 2t\right) =$$

$= T^2 + 2Tn - 3T - n^2 + n - \frac{2T^2}{n}\left(T-1\right) + \left(t-1\right)\left(2 - \frac{4T}{n}\right)$. Thus it follows that

$$
\begin{aligned}
LE(F_s) &= T^{'2} + T^{'} + (1 - \frac{2T^{'}}{n})2m_s \\
&= 2[T^2 - 2T + n - \frac{T^2}{n}(T-1) + (t-1)(1 - \frac{2T}{n})] \\
&= LE(F_t)\,.
\end{aligned}
$$

■

In the sequence, we construct pairs of L-equienergetic graphs where one belongs to family $\mathcal{G}_{\backslash,\mathcal{T}}$ and the other, to family $\mathcal{F}_{\backslash,\mathcal{T}}$.

Theorem 3.2. Consider the parameters $t^\flat$, $t^\sharp$ of Theorems 2.1 and t_1 and t_2 of Theorem 2.2. For all integer t with $1 \leq t \leq (n-T-1)(T-1)$, if $t \leq \min\{t^\flat, t_1\}$ or $t \geq \max\{t^\sharp, t_2\}$ then the graphs G_t in family $\mathcal{G}_{\backslash,\mathcal{T}}$ and F_t in family $\mathcal{F}_{\backslash,\mathcal{T}}$ are L-equienergetic.

Proof: As the graphs G_t and F_t have the same trace and same order n, the assertion follows from Theorems 2.1 and 2.2, since $LE(G_t) = T^2 + T + \left(1 - \dfrac{2T}{n}\right) 2m_t$ and $LE(F_t) = T^2 + T + \left(1 - \dfrac{2T}{n}\right) 2m'_t$. The constructions of the families guarantee that they have the same number of edges ($m_t = m'_t$) and different spectra.

■

The next corollary provides four non-isomorphic threshold graphs with same Laplacian energy.

Corollary 3.1. Consider the parameters $t^\flat$ and t_1 of Theorems 2.1 and of Theorem 2.2, respectively. For all integer t with $1 \leq t \leq (n-T-1)(T-1)$, if $t \leq \min\{t^\flat, t_1\}$ then the graphs G_t in family $\mathcal{G}_{\backslash,\mathcal{T}}$, G_s in family $\mathcal{G}_{\backslash,\mathcal{T}'}$, F_t in family $\mathcal{F}_{\backslash,\mathcal{T}}$ and F_s in family $\mathcal{F}_{\backslash,\mathcal{T}'}$ are L-equienergetic, where $T' = n - T$ and $s = (n - T' - 1)(T' - 1) - (t - 1)$.

Proof: As in Theorem 3.1, manipulating algebraically some inequalities, we verify that if $t \leq t_1(T)$ then $s \geq t^\sharp(T')$. In addition, by the proof of Theorem 3.1, we have $s \geq t_2(T')$. Then Theorem 3.2 assures that the graphs G_s and F_s, of families $\mathcal{G}_{\backslash,\mathcal{T}'}$ and $\mathcal{F}_{\backslash,\mathcal{T}'}$, respectively, are L-equienergetic. But from Theorem 3.1, we know that $LE(F_s) = LE(F_t)$ (F_t graph in family $\mathcal{F}_{\backslash,\mathcal{T}}$). By using again Theorem 3.2, we obtain $LE(F_t) = LE(G_t)$ (G_t graph of $\mathcal{G}_{\backslash,\mathcal{T}}$), finally proving that $LE(G_s) = LE(F_s) = LE(F_t) = LE(G_t)$.

■

Example 1. Considering $n = 12$ and $T = 7$ we have $T' = 5$. Taking $t = 4$ we have $LE(G_4) = LE(F_4) = LE(F_{21}) = LE(G_{21}) = 44$, where $G_4 \in \mathcal{G}_{\infty\in,\natural}$, $F_4 \in \mathcal{F}_{\infty\in,\natural}$, $F_{21} \in \mathcal{F}_{\infty\in,\bigtriangledown}$ and $G_{21} \in \mathcal{G}_{\infty\in,\bigtriangledown}$.

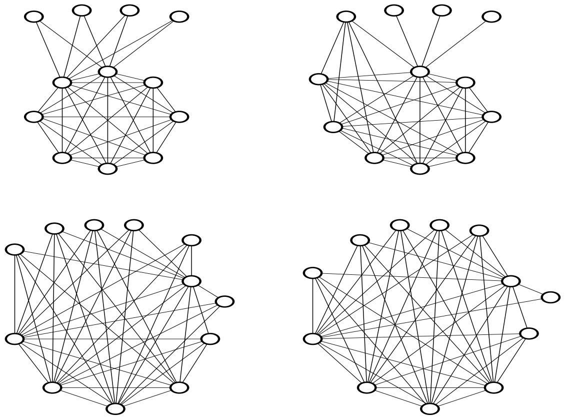

Figure 4: G_4, F_4, G_{21}, F_{21} with $LE(G_4) = LE(F_4) = LE(F_{21}) = LE(G_{21}) = 44$.

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