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New Results of the Geodeticity of the Contour of a Graph

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Abstract

A vertex v is called a *contour vertex* if no neighbor of v has an eccentricity greater than v. The *contour* $Ct(G)$ is the set formed by all contour vertices of G. Given a set $S \subseteq V(G)$, the *closed interval* $I[S]$ of S is the set formed by all vertices lying on shortest paths between any pair of vertices of S. We say that S is geodetic if $I[S] = V(G)$. We denote $I^2[Ct(G)] = I[I[Ct(G)]]$. In this work, we show structural and computational results for two problems proposed by Cáceres et al. 2005: *(i)* to determine whether the contour is geodetic; *(ii)* to determine if there exists a graph G such that $I^2[Ct(G)] \neq V(G)$. We contribute to the study of these problems by adding three infinite families of graphs whose contour is not geodetic to the set of only 3 known graphs since 2005. We prove that if $Ct(G) \subseteq S \subseteq V(G)$ and $|S| \geq |V(G)| - 3$, then S is geodetic. And so, when $S = Ct(G)$, we have a relation between $|V(G)|$ and the geodeticity of $I[Ct(G)]$. Using computational tools, we show that, if there exists a positive answer for the problem *(ii)* of Cáceres et al. 2005, the graph must contain at least 11 vertices.

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1 Introduction

We consider only finite, simple and connected graphs G. Given a set $S \subseteq V(G)$, we say that the *closed interval* $I[S]$ of S is the set of vertices lying on shortest paths between any pair of vertices of S. The set S is geodetic if $I[S] = V(G)$. The *distance* $d(v, w)$ between two vertices v and w is the number of edges in the shortest path between them. The *eccentricity* $ecc(v)$ of a vertex v is the maximum distance between v and any vertex w of G. A vertex v is a *contour vertex* if no neighbor of v has an eccentricity greater than v. The contour $Ct(G)$ of G is the set formed by all contour vertices of G [CMOLP05]. A vertex w is an *eccentric vertex* of some vertex v if the distance between v and w is equal to the eccentricity of v. We denote $I^2[S] = I[I[S]]$. The *diameter* $diam(G)$ of G is the maximum eccentricity of the vertices in $V(G)$. For more information about graph theoretical notation, see [BM08].

In 2005, Cáceres et al. [CMOLP05] proposed two questions concerning the contour of a graph: *(i)* the problem of determining whether the contour of a graph is geodetic; *(ii)* the problem of deciding if there exists a graph G such that $I^2[Ct(G)] = V(G)$.

The first graph whose contour is not geodetic was presented in 2005 in [CMOLP05]. A slight variation of this graph was presented in [CHM+08]. Only in 2013 [ADD+13] was exposed another graph whose contour is not geodetic.

Also in 2005, Cáceres et al. [CMOLP05] studied the contour of distance-hereditary graphs and showed that it is geodetic. Later, in 2008, Cáceres et al. [CHM+08] showed that the contour of every chordal graph is geodetic. In 2013, Artigas et al. [ADD+13] established a relation between the diameter of a graph and the geodeticity of its contour set. They proved that if $diam(G) \leq 4$, then $I[Ct(G)] = V(G)$ for every graph G. They also showed that if G is a bipartite graph such that $diam(G) \leq 7$, then $I[Ct(G)] = V(G)$.

In this work, we present structural properties for problem *(i)* of [CMOLP05].

We prove that for any set S, such that $Ct(G) \subseteq S \subseteq V(G)$ and $|S| \geq |V(G)| - 3$, we have that S is geodetic. Every graph G presented in the literature whose $Ct(G)$ is not geodetic is such that $|V(G) \setminus I[Ct(G)]| = 1$. Thus, our result implies that $I[Ct(G)]$ is geodetic for all of known examples. Here we present three infinite families of graphs whose contour is not geodetic, particularly, one of them is the first example such that $|V(G) \setminus I[Ct(G)]| > k$, $k > 0$. These three infinite families were obtained by variations of the previously known examples. We also prove that for integers $(i, j, k, l), i \geq 3$ and $j, k, l \geq 1$, there exists a graph with i contour vertices, j vertices that do not belong to $I[Ct(G)]$ and k contour vertices with l eccentric vertices which are not contour vertices. Finally, using computational tools, we verified that if $|V(G)| < 10$, then $Ct(G)$ is geodetic; and there exist only four non-isomorphic graphs with 10 vertices whose contour is not geodetic and we present these graphs. This directly implies that an answer for the problem *(ii)* of Cáceres et al. [CMOLP05] must contain at least 11 vertices.

2 Structural Results

In this section, we show structural results for the problem of determining whether the contour of a graph is geodetic. Before proving our main results, we present some others previously known.

Remark 2.1. If G is a graph and $v, u \in V(G)$, then $| ecc(v) - ecc(u) | \leq d(v, u)$. In particular, if $vw \in E(G)$ then $|ecc(v) - ecc(w)| \leq 1$.

Remark 2.2. Let G be a graph. If $e(v)$ is an eccentric vertex of $v \in V(G)$, then $ecc(e(v)) \geq ecc(v)$.

Lemma 2.1. [CHM+08] Let G be a graph and let $u_0 \in V(G)$. Suppose that $P = u_0, u_1, \ldots, u_t$ is a path in G such that $ecc(u_{i+1}) = ecc(u_i) + 1$, for each $i \in \{0, 1, \ldots, t-1\}$. Then, for each eccentric vertex $e(u_t)$ of u_t, there exists a geodesic between $e(u_t)$ and u_t that contains P. Further, $e(u_t)$ is an eccentric vertex of every vertex on P.

The next result establishes a relation between the cardinality of $Ct(G)$ and the problem of determining whether $Ct(G)$ is geodetic.

Theorem 2.2. *Let G be a graph and $S \subseteq V(G)$. If $Ct(G) \subseteq S$ and $|S| \geq |V(G)| - 3$, then S is geodetic.*

Proof. Let $|S| = |V(G)| - 1$ and $v_0 \notin S$. There exists a vertex v_1 adjacent to v_0 such that $v_1 \in S$ and $ecc(v_1) = ecc(v_0) + 1$. Since $ecc(v_1) > ecc(v_0)$, there exists a vertex x adjacent to v_0 such that x is not adjacent to v_1. Consequently, $v_0 \in I[x, v_1] \subseteq I[S]$.

Let $|S| = |V(G)| - 2$ and $x, y \notin S$. If x and y are not adjacent, we apply the previous argument twice. Consider that x and y are adjacent. If $ecc(x) + 1 = ecc(y)$, since $y \notin Ct(G)$, there exists a vertex z adjacent to y such that $ecc(y) + 1 = ecc(z)$. Let $e(z)$ be an eccentric vertex of z, by Lemma 2.1 there exists a geodesic between z and $e(z)$ containing x and y. Since $ecc(e(z)) \geq ecc(z)$, we conclude that $e(z) \neq x$, $e(z) \neq y$ and $e(z) \in S$. Consequently, $x, y \in I[S]$. If $ecc(x) = ecc(y)$, then there exists a vertex z adjacent to x such that $ecc(z) = ecc(x) + 1$. Let $e(z)$ be an eccentric vertex of z , by Lemma 3 there exists a geodesic between z and $e(z)$ containing x. Since $ecc(e(z)) \geq ecc(z)$, we conclude that $x \in I[S]$. Analogously, we conclude that $y \in I[S]$.

Let $|S| = |V(G)| - 3$ and $x, y, z \notin S$. If $\{x, y, z\}$ is an independent set then we use the argument of the case $|S| = |V(G)| - 1$. Consider that x, y are adjacent and z is not adjacent to x or y. We separately analyse the vertex z and vertices x, y, the arguments are similar to the previous ones. Now, suppose that $\{x, y, z\}$ is a clique. By Remark 2.1, at least two of them have the same eccentricity. Without loss of generality, consider that $ecc(x) = ecc(y)$ and $ecc(x) = ecc(z) - 1$. Let v be a vertex adjacent to z such that $ecc(v) = ecc(z) + 1$, therefore v has an eccentric vertex $e(v)$ such that, by Remark 2.2, $ecc(e(v)) \geq ecc(v)$ and $e(v) \in S$, which implies that there exists a geodesic between v and $e(v)$ that contains x, z and y, z. Hence, $\{x, y, z\} \subseteq I[S]$. The case that $ecc(x) = ecc(z) + 1$ is similar to the previous one. It remains to consider the case where $\{x, y, z\}$ induces a P_3, and the arguments are analogous to those used in this proof. ∎

We contribute to the problem of determining whether the contour of a graph is geodetic by generalizing two graphs found in the literature (see Figure 1). We construct three infinite families of graphs whose contour is not geodetic. We refer to Figure 2 and Figure 3.

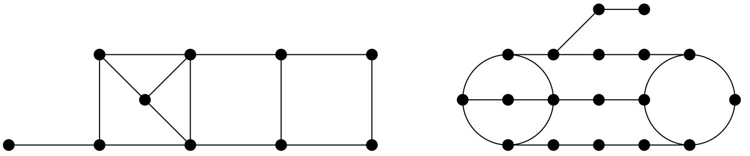

***Figure 1**: graphs whose contour is not geodetic [ADD+13, CHM+08]*

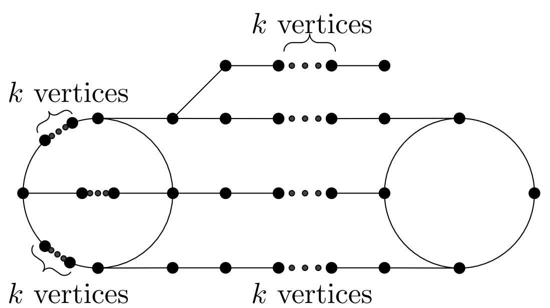

***Figure 2:** First infinite family of graphs whose contour is not geodetic.*

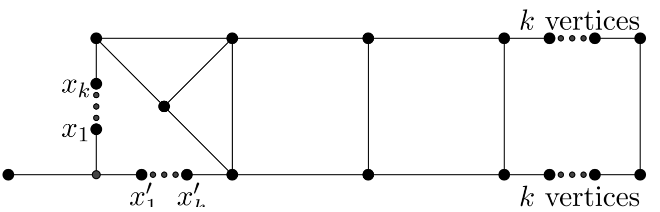

***Figure 3:** Second infinite family of graphs whose contour is not geodetic.*

The third family can be obtained from the family presented in Figure 3, by removing either one vertex x_k or one vertex x'_k. The graphs presented in the literature whose contour is not geodetic are such that $|V(G) \setminus$

$I[Ct(G)]| = 1$. We see that the infinite family shown in Figure 2 is such that $|V(G) \setminus I[Ct(G)]| = k$, $k > 0$. $I[Ct(G)]$ is geodetic for the three families, i.e. $I^2[Ct(G)] = V(G)$.

The following theorem guarantees that it is possible to construct graphs with any number of contour vertices and any vertices that are not in $I[Ct(G)]$.

Theorem 2.3. *For any integers (i, j, k, l) such that $i \geq 3$, $j, k, l \geq 1$, there exists a graph G with i contour vertices, j vertices that does not belong to $I[Ct(G)]$ and k contour vertices with l eccentric vertices which are not contour vertices.*

3 Computational Results

In this section we show computational results obtained in this research. We start this approach by searching for all graphs whose contour is not geodetic in the set of all graphs with a fixed number of vertices. In the first step we generate all non isomorphic graphs with a fixed number of vertices and then we implement and execute the Algorithm 1 to verify whether the contour is geodetic. Hereinafter we present our main algorithm.

Algorithm 1 Algorithm to check whether $I[Ct(G)] = V(G)$.

1: Calculate the eccentricity of every vertex of G;
2: Determine for each vertex v if $v \in Ct(G)$;
3: Verify if $I[Ct(G)] = V(G)$.

Applying this algorithm, we obtain the next results. They completely classify the problem for all graphs with up to 10 vertices.

Theorem 3.1. *If G is a graph with at most 9 vertices, then $I[Ct(G)] = V(G)$.*

Theorem 3.2. *There exists four graphs G with 10 vertices such that $I[Ct(G)] \neq V(G)$. Moreover, the graphs are depicted in Figure 4.*

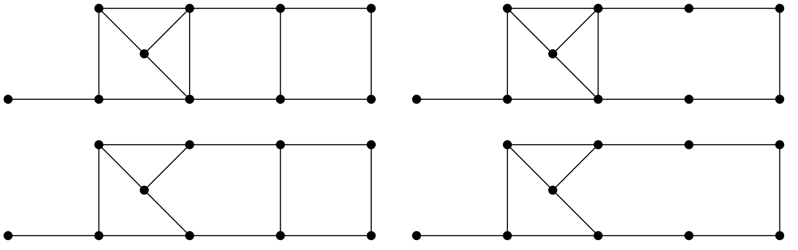

Figure 4: *The unique four graphs with 10 vertices whose contour is not geodetic.*

Corollary 3.3. If G is a graph with at most 10 vertices, then $I^2[Ct(G)] = V(G)$.

Proof. The Theorems 3.1 and 3.2 guarantee that the only graphs with up to 10 vertices whose contour is not geodetic are the ones shown in Figure 4. For all of them, $I^2[Ct(G)] = V(G)$. ■

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