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Total coloring of snarks is NP-complete

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Abstract

Snarks are bridgeless cubic graphs that do not allow 3-edge-colorings. We prove that the problem of determining if a snark is of Type 1 is NP-complete.

1 Introduction

Let $G = (V, E)$ be a finite 3-regular graph with vertex set V and edge set E. A *k-total-coloring* of G is an assignment of k colors to the edges and vertices of G, so that adjacent or incident elements have different colors. The *total chromatic number* of G, denoted by $\chi''(G)$, is the least k for which G has a k-total-coloring. The well-known Total Coloring Conjecture states that $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$ (where $\Delta(G)$ is the maximum degree of G) and it has been proved for cubic graphs [Ros71]. Hence, the total chromatic number of a cubic graph is either 4, in which case the graph is called *Type 1*, or 5, in which case it is called *Type 2*. *Snarks* are bridgeless cubic graphs that do not allow 3-edge-colorings (Class 2), and their importance arises at least in part from the fact that several well-known conjectures would have snarks as minimal counterexamples. Some common definitions used in this paper will be omitted due to space constraints.

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In 2003, Cavicchioli et al. verified that all snarks with girth at least 5 and fewer than 30 vertices are Type 1 [CMRS03]. In 2011, Campos et al. proved that the infinite families of Flower and Goldberg snarks are Type 1 [CDdM11]. In 2013, Brinkmann et al. verified that all snarks with such girth and fewer than 38 vertices are Type 1 [BGHM13]. Later on, Sasaki et al. proved that both Blanuša families and a part of Loupekine family are Type 1 and presented some Type 2 snarks with small girth [SDdFP14]. Motivated by the question proposed by Cavicchioli et al. [CMRS03] of finding, if one exists, the smallest Type 2 snark of girth at least 5, we investigate the total coloring of snarks.

It is shown in [SA89] that the problem of determining if a cubic bipartite graph is Type 1 is NP-complete. We prove that, similarly, the problem of determining if a snark is Type 1 is NP-complete. Our proof resembles the one in [SA89] but requires a slightly different construction. The proof is by reduction from the well-known NP-complete problem of determining if a 4-regular graph has a 4-edge-coloring (Class 1).

Preliminaries Since this work is based on the proofs of the NP-completeness of the problem of deciding whether a bipartite cubic graph is Type 1 [SA89] and has an equitable 4-total-coloring [DdFM+16], we start by presenting useful coloring properties determined in both papers.

Lemma 1 (Sanchez-Arroyo [SA89])**.** In each 4-total-coloring of K (resp. H) the three (resp. four) pendant edges of K (resp. H) receive the same color (see Figure 1).

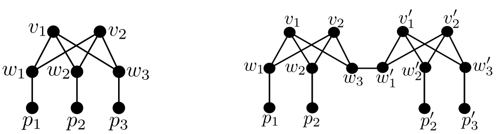

Figure 1: Graphs K and H, respectively.

Lemma 2 (Dantas et al. [DdFM+16]). Consider any proper partial 4-coloring C^P of H such that only w'_3, all pendant edges and all pendant vertices are colored, and:

- all the pendant edges have the same color, say i,
- p_1, p_2 have distinct colors, say resp. j and k (see Figure 2).

This coloring C^P can be extended to the vertices $w_1, w_2, w_3, w'_1, w'_2$ and edge $w_3w'_1$ so that it is still proper and the colors of w_1, w_2, w_3 (resp. w'_1, w'_2, w'_3) are all distinct.

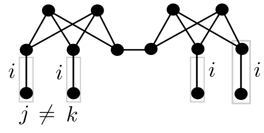

Figure 2: The framed elements are already colored by the proper partial 4-coloring C^P.

Lemma 3 (Dantas et al. [DdFM+16]). Consider a proper partial 4-coloring of the pendant edges of K and their extremities, such that all pendant edges are colored with the same color and w_1, w_2, w_3 are colored with the three other colors. This coloring may be extended to a 4-total-coloring of K (see Figure 3).

As a corollary of Lemmas 2 and 3, we obtain the result that any partial coloring satisfying the conditions of Lemma 2 can be extended to a 4-total-coloring of H.

In this work, we prove the following result on the total coloring of snarks.

Theorem 1.1. *The problem of deciding whether or not a snark is Type 1 is NP-complete.*

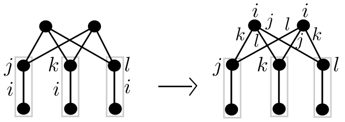

Figure 3: An extension of a proper partial 4-coloring of K satisfying the hypothesis of Lemma 3. The framed elements are already colored by a proper partial 4-coloring.

2 Proof of Theorem 1.1

Since we can verify in polynomial time that a candidate coloring is a 4-total-coloring, the problem is in the class NP.

Given a 4-regular graph G, we construct a snark G^R by replacing each vertex of G by the graph R. The graph R is obtained from four disjoint copies of the graph H and two disjoint copies of the Petersen graph P and due to the construction of R, it preserves interesting coloring properties of H. In the following, we prove that the graph G has a 4-edge-coloring if and only if the snark G^R has a 4-total-coloring.

Construction of graph G^R from G Let G be a 4-regular graph. A graph G^R is built as follows. G^R contains a disjoint copy R_v of R, for each vertex v of G. Two copies of R are joined by an edge whenever the corresponding vertices are adjacent in G, so that there is a one-to-one correspondence between the set of edges of G and the set of edges of G^R that connect two copies of R. We call the edges connecting copies of R *connecting edges* of G^R. The construction of G^R can clearly be done in polynomial time in the order of G.

We denote by R^4, the graph R plus the 4 connecting edges and their respective endvertices shown in Figure 4.

The next two results are similar to the ones in Dantas et al. [DdFM$^+$16] since our construction preserves the key coloring properties used to prove the corresponding results in that paper.

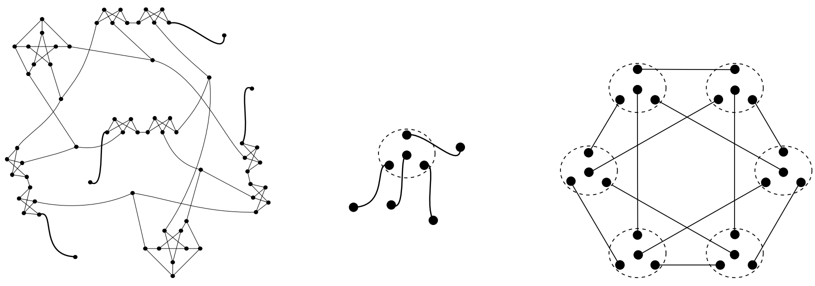

Figure 4: The graph R^4 on the left, a representation of it in the middle, and a depiction of the graph G^R obtained from the 4-regular graph G on 6 vertices and 12 edges on the right.

Claim 1. If G^R is Type 1, then G is Class 1.

Proof of Claim 1. Suppose that there exists a 4-total-coloring C^T of G^R, and let us consider the 4-total-coloring induced by C^T on R^4_v for any vertex v of G. By the construction of R^4, since any two of the four copies of H contained in R^4_v have adjacent pendants, we obtain that C^T assigns four distinct colors to the connecting edges incident to R_v. So, assigning to each edge vw of G the color given by C^T to the connecting edge between R_v and R_w we obtain a 4-edge-coloring of G. ∎

Claim 2. If G is Class 1, then G^R is Type 1.

Proof of Claim 2. Let C^E be a 4-edge-coloring of G. Starting from this coloring we will define a 4-total-coloring C^T of G^R. We define first the colors of the connecting edges of G^R: for every edge vw of G we assign the color $C^E(vw)$ to the corresponding connecting edge E_{vw} of G^R. Then, we assign colors to the extremities of the connecting edges with any two available distinct colors. At this moment, the coloring is a proper partial 4-coloring of G^R that assigns, in each copy of R^4, colors to all pendant edges and their extremities. For a vertex v of G, let the four connecting edges incident to R^4_v be colored i, j, k, l as on Figure 5. In this figure, we show how this coloring can be extended to a proper coloring of all edges and vertices of R^4 that are not inside a $K_{2,3}$ of a copy of H. Doing this for every copy of R^4, we extend the present coloring to a proper partial 4-coloring of G^R that colors the extremities of the connecting edges, and all other vertices and edges of G^R that are not inside a copy of H.

Noticing that the proper partial 4-coloring on Figure 5 is such that the conditions of Lemma 2 are verified for every copy of H in R^4, and since it colors every copy of R^4 as in Figure 5, we can apply Lemmas 2 and 3 in order to extend the coloring to a 4-total-coloring C^T of G^R. ■

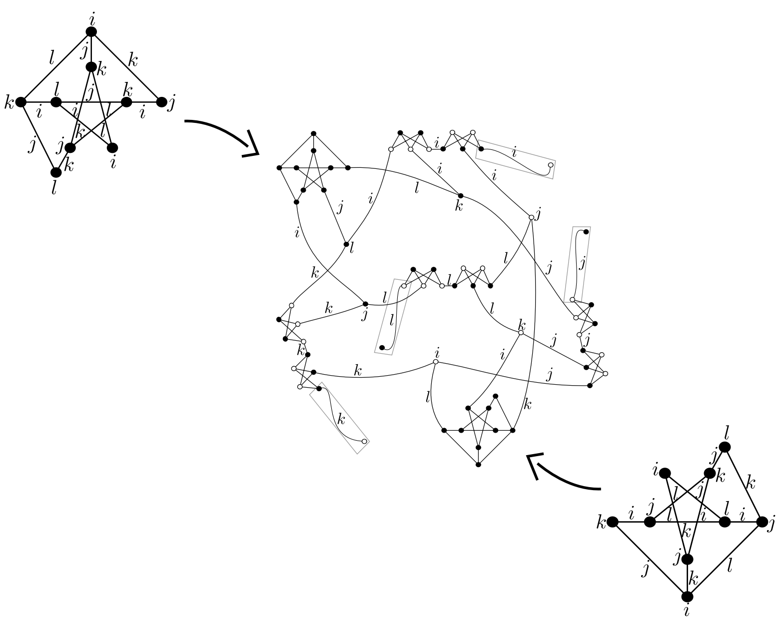

Figure 5: An extension of a proper partial 4-coloring of the framed elements of R^4.

It remains to show that the constructed graph G^R is a snark.

Definition 2.1 (Isaacs, 1975 [Isa75])**.** *Given a cubic graph G and a vertex x of G, the cubic semi-graph obtained by removing vertex x will be denoted by G_x. Given two cubic graphs G and H, any cubic graph obtained from G_x and H_y, for some vertices x and y, by connecting the semi-edges of G_x*

to the semi-edges of H_y is said to be obtained by a 3-construction *from G and H (see Figure 6).*

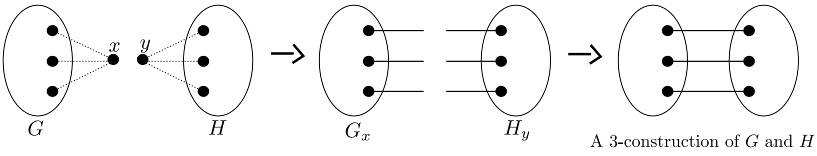

Figure 6: Graphs G and H and a 3-construction of G and H.

Lemma 4 (Isaacs, 1975 [Isa75])**.** If a cubic graph F, obtained by the 3-construction of bridgeless cubic graphs G and H, such that at least one of G or H is a snark, then F itself is also a snark.

Let G^{R-} be the cubic graph obtained from G^R by replacing each Petersen graph by a vertex in all copies of R. The graph G^R is obtained by $2|V(G)|$ 3-constructions of the Petersen graph and G^{R-}. Since the Petersen graph is a snark, the graph G^R is a snark. This ends the proof of Theorem 1.1.

3 Final considerations

Let A be a proper subset of V. We call the set F of edges of G with one endpoint in A and the other endpoint in $V \setminus A$, *the edge cutset induced by A*. A subset F of edges of G is *an edge cutset* if there exists a proper subset A of V such that F is the edge cutset induced by A. If $G[A]$ and $G[V \setminus A]$ contain cycles, then F is said to be a *c-cutset*. We say that G is *cyclically k-edge-connected* if it does not have a c-cutset of cardinality smaller than k. If G has at least one c-cutset, the *cyclic-edge-connectivity* of G is the smallest cardinality of a c-cutset of G.

There are many definitions of snarks in the literature and the one most used is cyclically-4-edge-connected cubic graphs of Class 2. In this work, we consider snarks simply as bridgeless cubic graphs of Class 2 and prove that the problem of determining whether a snark is Type 1 is NP-complete. More precisely, our proof holds for snarks with cyclic-edge-connectivity 3, since the smallest cardinality of a c-cutset of the constructed graph G^R is 3. Indeed, for cyclic-edge-connectivity 1, 2 or 3 there exist examples of Class 2 cubic graphs of each Type [SDdFP14].

Cyclically-4-edge-connected cubic graphs of Class 2 and Type 2 have recently been found [BPS15] (all containing squares). So, also for cyclic-edge-connectivity 4 there exist examples of Class 2 cubic graphs of each Type [BPS15, SDdFP14]. In order to investigate the complexity problem of determining whether a cyclically-4-edge-connected cubic graph of Class 2 is Type 1, another approach is necessary, since our gadget has several c-cutsets of size 3. We leave this as an open problem.

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