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The Burning of the Snark

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Abstract

The firefighter game is a model of the containment of the spreading of an undesired property within a network, like an infecting disease. In 2007, Finbow et al. showed that finding an optimal strategy is NP-hard for trees of maximum degree three, and presented a tractable case on graphs of maximum degree three when the fire breaks out at a vertex of degree two. This implies that the firefighter game is hard for graphs of maximum degree three such that the fire breaks out in a vertex of degree three. So, a natural question arises: Is there a subclass of graphs of degree at most three for which the optimal strategy can be computed efficiently? In this paper, we show how to determine optimal strategies for Blanusa, Flower, and Goldberg snarks. We calculate their surviving rate, which is average proportion of vertices that can be saved.

1 Introduction

The Firefighter game was introduced by Hartnell at the 25th Manitoba Conference on Combinatorial and Computing in Winnipeg (1995). It is a model of the containment of the spreading of an undesired property within a network, like an infecting disease. Let (G, v) be a pair where G

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is a simple, undirected and connected graph, and v is a specified vertex of G, the root of G. The game proceeds in rounds. At round 0, a fire breaks out at vertex v. In subsequent rounds, the firefighter *defends* at most one vertex, which is not burning and was not defended in previous rounds; and the fire spreads to all vertices of G that are neither burning nor defended and have a burning neighbour. Once burning or defended, a vertex remains so for the rest of the game. The process ends when the fire can spread no further. The objective is to build a *strategy* in order to minimise the damage on the graph, that is, the player chooses a sequence of vertices for the firefighter to protect, so as to burn the minimum number of vertices on the graph.

FIREFIGHTER

Instance: A rooted graph (G, v) and an integer $k \geq 1$.

Question: If the fire breaks out at v, is there a strategy under which at most k vertices burn on graph G?

Let $\text{sn}(G, v)$ denote the maximum number of vertices of G that the firefighter can save when the fire breaks out at vertex v. When there is no ambiguity, we write only $\text{sn}(v)$, and we emphasise that generally the parameter $\text{sn}(G, v)$ depends heavily on v. Let $\rho(G, v) = \text{sn}(G, v)/n$ be the proportion of vertices saved where n denote the order of G. The *surviving rate* $\rho(G)$ of a graph G with order n is defined to be the average proportion of vertices that can be saved when a fire breaks out at a random vertex of the graph, i.e.,

$$\rho(G) = \frac{1}{n} \sum_{v \in V} \rho(G, v) = \sum_{v \in V} \frac{\text{sn}(v)}{n^2}.$$

MacGillivray et al. [MW03] showed that FIREFIGHTER is NP-complete even if G is bipartite. Finbow et al. [FKMR07] showed that FIREFIGHTER is NP-complete for trees of maximum degree three, and presented a tractable case on graphs of maximum degree three when the fire breaks out at a vertex of degree two. This implies that the FIREFIGHTER problem is NP-complete for graphs of maximum degree three such that the fire breaks out in a vertex of degree three. Even with respect to approximation algorithms for the FIREFIGHTER problem and its variants, there exist only few known results on trees [CVY08, CDD+13, HL00] and graphs of bounded treewidth [CCVZ10]. Given the difficulty of the FIREFIGHTER problem, it is natural to study this problem for graph classes: outerplanar graphs [WYZ11]; interval, split, permutation, and P_k-free graphs [FHvL12]; planar graphs [KWZ12]; square girds and hexagonal grids [GKP14].

Here we study the FIREFIGHTER problem on some well-known snarks. Snarks are simple connected bridgeless cubic graph whose edges cannot be properly coloured with three colours. The name snark was given by Gardner in 1976 and was based on the poem by Lewis Carroll "The Hunting of the Snark". The definition of *snarks* was motivated by the search of counter-examples to the four-colour conjecture. The importance of these graphs stems so far from the fact that several relevant conjectures stated in the past would have snarks as minimal counter-examples: Tutte's 5-Flow Conjecture, the 1-Factor Double Cover Conjecture, and the Cycle Double Cover Conjecture. In this work, we contribute to the study of FIREFIGHTER by presenting an algorithm that returns optimal strategies for Blanusa, Flower and Goldberg snarks [Bla46, CDdM11, Gol81, Isa75, SDdF11] and we calculate their surviving rates.

2 Main results

Our results are based on work of Fomin et al. [FHvL12] who showed an optimal strategy for interval graphs using the idea of surrounding the fire by a special vertex set. Since Blanusa, Flower, and Goldberg snarks [Bla46, CDdM11, Gol81, Isa75, SDdF11] are constructed using basic blocks, we use a "sufficient" number of blocks, called *container subgraph*, and prevent the fire from spreading to the graph by defending "extreme" vertices of the container subgraph. In our implemented algorithm we investigate all possible strategies on the container subgraph, which yields an overall optimal strategy for Blanusa, Flower and Goldberg snarks.

Throughout this section, we consider a snark G, with vertex set $V(G)$ and edge set $E(G)$. Furthermore, we consider $V(G) = V(B_0) \cup V(B_1) \cup \ldots \cup V(B_{l-1})$, where each B_i is a basic block, $0 \leq i \leq l-1$, with l sufficiently large. We remark that these snarks have the property that there are no edges between B_i and B_j, $j - i$ more than 2. Given a set S and a property Π, we say that S is *minimal* with respect to Π, if no proper subset of S has property Π. Let $u, v \in V(G)$. A (u, v)-*path* is a sequence of distinct vertices $u_0 u_1 \ldots u_k$ such that u_i is adjacent to u_{i+1}, $0 \leq i \leq k-1$, for $u_0 = u$, and $u_k = v$. Let $u, v \in V(G)$ be non-adjacent. A set $S \subseteq V(G)$ is called (u, v)-*separator*, if u and v belong to different components of $G - S$. The vertex $u \in B_i$ is a *link-vertex* if it has a neighbour in B_{i-1} or B_{i+1}. Let $u \in V(B_i)$, $v \in V(B_j)$, $i \neq j$, and let k be the number of edges that join block B_j to blocks B_{j-1} and B_{j+1}. It is possible to obtain a minimal (u, v)-separator of size k, by choosing all link-vertices of $B_{j-1} \cup B_{j+1}$ that have a neighbour in $V(B_j)$. We denote by S_v a *minimal* (u, v)-*separator* with cardinality k as defined previously such that $d_G(u, v) \geq d_G(w, v) \geq k + 1$ for all $w \in S_v$. Since S_v has order k and the fire needs at least $k + 1$ rounds to burn any vertex of S_v, if $\overline{\sigma}$ is a strategy that defends all vertices of S_v in the first k rounds, then $\overline{\sigma}$ surrounds the fire. Let $C_{\overline{\sigma}}$ be the component of $G[V(G) \setminus S_v]$ that contains u. Note that $\overline{\sigma}$ saves all vertices of $C_{\overline{\sigma}}$. We call S_v a set of *extreme vertices* and the induced subgraph $G[(V(G) \setminus V(C_{\overline{\sigma}})) \cup S_v]$ a S_v-*container subgraph.*

Lemma 1. If σ_o is an optimal strategy for the S_v-container subgraph such that the fire starts at v and saves all vertices of S_v, then strategy σ_o is an optimal strategy for G.

Proof. Let $\overline{\sigma}$ be any strategy such that S_v is a set of defended vertices, let $C_{\overline{\sigma}}$ be the component that contains u in $G[V(G) \setminus S_v]$, and let $w \in C_{\overline{\sigma}}$. Since S_v is (u, v)-separator, each (v, w)-path has a vertex of S_v. By hypothesis, σ_o saves all vertices of S_v in the S_v-container subgraph. So each

(v, w)-path has a vertex defended by σ_o, and therefore, the vertex w is saved by strategy σ_o. Since σ_o is an optimal strategy for the S_v-container subgraph and it is a strategy for G that saves all vertices of $V(C_{\overline{\sigma}}) \cup S_v$, we have that σ_o is an optimal strategy for the whole graph G. □

We remark that if there exists a strategy σ for the S_v-container subgraph that burns at least one vertex of S_v, then we cannot conclude that σ is an optimal strategy for the whole graph G. The DEFENCE TEST algorithm was implemented and executed on the S_v-container subgraph and it investigates all possible strategies. By the symmetry of the snarks and Lemma 1, finding optimal strategies for these infinite classes of graphs reduces to a finite problem, which we solved by exhaustive search. Each defence is a t-permutation $P(n_v, t)$ of the vertex set of the S_v-container subgraph, where n_v is the order of the S_v-container subgraph and t is the number of vertices defended by the strategy during the game.

First, we apply the DEFENCE TEST algorithm on Blanusa snarks BF_l and present its surviving rate. We refer to Figure 1 for an example of the first Blanusa snark BF_5.

Lemma 2. Let BF_l, $l \geq 5$, be a Blanusa snark. A single fire starting at vertex $v \in \{x_i, z_i, r_i, t_i\}$, with $0 \leq i \leq l-1$, can be contained by one firefighter per round in four rounds, and the minimum number of burned vertices is seven.

Proof. We refer to Figure 1 for the notation applied. The number of edges that join one basic block to its adjacent blocks in BF_l, $l \geq 5$, is four, therefore $|S_v| = 4$ for all vertex $v \in V(BF_l)$. If $v = x_0$, we consider $S_{x_0} = \{t_1, r_1, r_{l-1}, t_{l-1}\}$. Note that the distances $d_{BF_l}(x_0, t_1) = d_{BF_l}(x_0, r_{l-1}) = 5$, and $d_{BF_l}(x_0, r_1) = d_{BF_l}(x_0, t_{l-1}) = 6$. Considering $\overline{\sigma} = (t_1, r_1, r_{l-1}, t_{l-1})$, we have $C_{\overline{\sigma}} = \{r_1, t_1\} \cup B_2 \cup \ldots \cup B_{l-1}$, that is, the set S_{x_0}-container subgraph is equal to $B_0 \cup B_1 \cup B_{l-1} \cup \{r_{l-1}, t_{l-1}\}$.

Applying the DEFENCE TEST algorithm for (S_{x_0}-container subgraph, x_0), it returns the strategy $\sigma_{DT} = (a, r_0, t_0, r_l)$. Thus, all vertices of S_{x_0} are saved by σ_{DT} and, by Lemma 1, the strategy σ_{DT} is an optimal strategy for BF_l. The total number of burned vertices given by σ_{DT} is seven. By similar arguments, the algorithm obtains optimal strategies for each vertex x_i, z_i, r_i, t_i with $0 \leq i \leq l-1$. □

We present in Table 1 the corresponding results for other fire sources in BF_l.

Theorem 1. The surviving rate ρ of Blanusa snark BF_l is $1 - \frac{64l+47}{2(5+4l)^2}$, $l \geq 5$.

Proof. Since $|V(BF_l)| = 10+8l$, by Lemma 2 and similar results for other fire sources (Table 1), $0 \leq i \leq l-1$ and $1 \leq j \leq l-1$, we obtain:

$$\begin{cases} \text{sn}(BF_l, x_i) = \text{sn}(BF_l, z_i) = \text{sn}(BF_l, r_i) = \text{sn}(BF_l, t_i) & = 3 + 8l \\ \text{sn}(BF_l, a) = \text{sn}(BF_l, b) = \text{sn}(BF_l, u_0) = \text{sn}(BF_l, v_0) = \text{sn}(BF_l, s_0) & = 2 + 8l \\ \text{sn}(BF_l, u_j) = \text{sn}(BF_l, v_j) = \text{sn}(BF_l, y_j) = \text{sn}(BF_l, s_j) & = 1 + 8l \\ \text{sn}(BF_l, y_0) & = 8l \end{cases}.$$

Thus,

$$\begin{aligned} \rho(BF_l) &= 4\sum_{i=0}^{l-1} \frac{3+8l}{(10+8l)^2} + 5\left(\frac{2+8l}{(10+8l)^2}\right) + 4\sum_{j=1}^{l-1} \frac{1+8l}{(10+8l)^2} + \frac{8l}{(10+8l)^2} \\ &= \frac{1}{(10+8l)^2}\left[(8l)^2 + 32l + 6\right] = 1 - \frac{64l+47}{2(5+4l)^2}. \end{aligned}$$

□

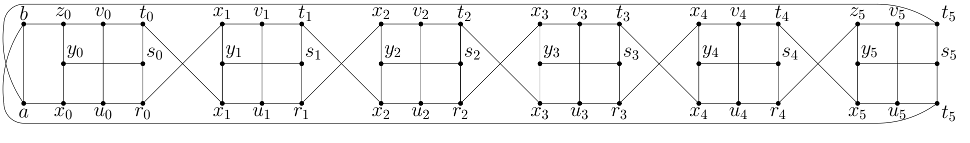

Figure 1: Blanusa Snark BF_5.

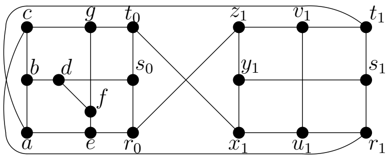

Figure 2: Blanusa Snark BS_1.

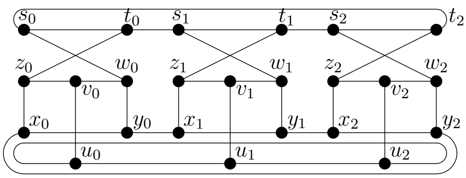

Figure 3: Goldberg Snark G_3.

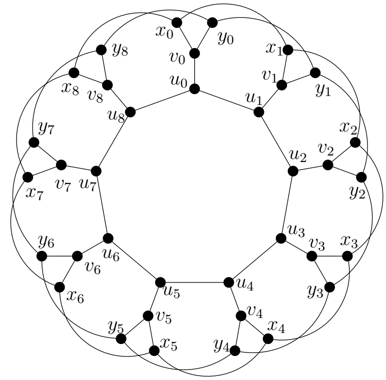

Figure 4: Flower Snark F_9.

The same analysis was performed for Blanusa BS_l, Flower F_l, and Goldberg G_l snarks, (Figures 2, 3, and 4), and we summarise our results in Table 1.

Snark G	$\|V(G)\|$	Fire source	Number of rounds	Total burned vertices	$\rho(G)$
BF_l	$10+8l$	$x_i, z_i, r_i,$ or t_i	4	7	$1-\frac{64l+47}{2(5+4l)^2}$
		$a, b, u_0, v_0,$ or s_0	4	8	
		$u_j, v_j, y_j,$ or s_j	5	9	
		y_0	5	10	
BS_l	$10+8l$	$x_j, z_j, r_j,$ or t_j	4	7	$1-\frac{32l+25}{(5+4l)^2}$
		a, b, c, e, g, r_0, s_0 or t_0	4	8	
		$u_j, v_j, y_j,$ or s_j	5	9	
		d or f	5	10	
F_l	$4l$	$u_i, x_i,$ or y_i	6	14	$1-\frac{61}{16l}$
		v_i	6	19	
G_l	$8l$	v_i	6	12	$1-\frac{57}{32l}$
		$x_i, y_i, z_i, w_i, s_i,$ or t_i	6	14	
		u_i	8	18	

Table 1: The indices $0 \le i \le l-1$ and $1 \le j \le l-1$ are related to label blocks.

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