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On the diameter of the Cayley Graph $H_{\ell,p}$

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Abstract

The family $H_{\ell,p}$ has been defined in the context of edge partitions. Subsequently, it was shown to be composed of Hamiltonian Cayley graphs, and that it is possible to determine the diameter of $H_{\ell,p}$ in $O(\ell)$ time. The established properties such as the low diameter suggest the $H_{\ell,p}$ graph as a good topology for the design of interconnection networks. The $p^{\ell-1}$ vertices of the graph $H_{\ell,p}$ are the ℓ-tuples with values between 0 and $p-1$, such that the sum of the ℓ values is a multiple of p, and there is an edge between two vertices if the two corresponding tuples have two pairs of entries whose values differ by one unit. Our goal is to find the diameter of Cayley graph $H_{\ell,p}$ in time $O(log(\ell+p))$. In this work, we did this for some families of graphs. We also show that the diameter of $H_{\ell,p}$ is the same of $H_{p,\ell}$. Finally, we find a tight upper bound on the diameter of $H_{\ell,p}$.

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1 Introduction

In this work, our interests are the design and analysis of static networks. Static networks can be modeled using tools from Graph Theory. A graph represents an interconnection network, where the processors are the vertices and the communication links between processors are the edges connecting the vertices. There are several parameters of interest to specify a network: low degree, low diameter, and the distribution of the node disjoint paths between a pair of vertices in the graph. The degree relates to the port capacity of the processors and hence to the hardware cost of the network. The maximum communication delay between a pair of processors in a network is measured by the diameter of the graph. Thus, the diameter is a measure of the running cost. The number of parallel paths between a pair of nodes is limited by the degree of the underlying graph, the knowledge of this distribution is helpful in the evaluation of the fault-tolerance of the network [Kon08, VS96].

The definition of Cayley graphs was introduced to explain the concept of abstract groups, which are described by a generating set. Cayley graphs are regular, in some cases, have logarithmic diameter and can be used to design interconnection networks [LJD93, VS96]. One graph commonly used to design interconnection networks is the "hypercube graph". The class $H_{\ell,p}$ not only are Cayley graphs but also are Hamiltonian as we can see in [dCRdFK10]. There also exist an algorithm to calculate the diameter of graph $H_{\ell,p}$ of time $O(\ell)$ [dCRdFK12]. In this work, we find a formula which can be evaluated in time $O(log(\ell+p))$ for the diameter for some families of graphs. We also show that the diameter of $H_{\ell,p}$ is the same of $H_{p,\ell}$. Finally, we find a tight upper bound for the diameter of $H_{\ell,p}$.

2 Cayley graphs

In this section, we define a family of Cayley graphs denoted by $H_{\ell,p}$. First, we give some definitions about Cayley graphs.

Definition 1. Let G be a group and $C \subseteq G$. We say that C is a *generating set* of G, if any element of G can be obtained from elements of C by a finite number of applications of the operation $+$.

Now, we define the concept of Cayley graph.

Definition 2. We say that a directed graph $\Gamma = (V, E)$ is a *Cayley graph* for a group $(G, +)$ with a generating set C, if there is a bijection mapping every $v \in V$ to an element $g_v \in G$, such that (v, w) is a directed edge of E if and only if there exists $c \in C$ such that $g_w = c + g_v$.

If the identity element $\iota \notin C$, then there are no loops in Γ, hence we say that Γ satisfies the *identity free* property. If $c \in C$ implies $-c \in C$, then for every edge from g to $g + c$, there is also an edge from $g + c$ to $(g + c) + (-c) = g$, for this reason we say that Γ satisfies the *symmetry condition*. A Cayley graph that satisfies both the identity free property and the symmetry condition is an undirected graph. In this paper, we only consider these graphs.

Lemma 3. *[Kon08]* Cayley graphs are vertex transitive.

2.1 The $H_{\ell,p}$ family

In this section, we define the group $(V_{\ell,p}, +)$ and the generating set $C_{\ell,p}$. Subsequently, we show that the graph $H_{\ell,p}$ is the Cayley graph associated to the group $(V_{\ell,p}, +)$ with generating set $C_{\ell,p}$.

For each $\ell \geq 2$ and $p \geq 2$, Holyer [Hol81] defines a graph $H_{\ell,p} = (V_{\ell,p}, E_{\ell,p})$ where

$V_{\ell,p} = \{x = (x_1, ..., x_\ell) \in Z_p^\ell \text{ with } \sum_{i=1}^{\ell} x_i = 0 \bmod p\}$,

$E_{\ell,p} = \{(x, y) :$ there are distinct i, j such that $y_k \equiv_p x_k$ for $k \neq \{i, j\}$

and $y_i \equiv_p x_i + 1, \;\; y_j \equiv_p x_j - 1$ or $y_i \equiv_p x_i - 1, \;\; y_j \equiv_p x_j + 1\}$.

An $H_{\ell,p}$ graph consists of vertices that contain ℓ components with values between 0 and $p - 1$, such that the sum of its components is a multiple of p, with $p \in \mathbb{Z}_+$.

The vertices of $H_{\ell,p}$ are the elements of a finite group $(V_{\ell,p},+)$, where the operation $+$ is such that $(f_{a_1},\dots,f_{a_\ell})+(f_{b_1},\dots,f_{b_\ell})=(f_{a_1}+f_{b_1},\dots,$ $f_{a_\ell}+f_{b_\ell})$, with $f_{a_i},\ f_{b_i}\in(Z_p,+)$. The pair $(V_{\ell,p},+)$ is a finite group. The element $(0,0,\dots,0,0)\in V_{\ell,p}$ is the identity element.

Definition 4. We define $C_{\ell,p}=\{e_{i,j}\}$ as the set of l-tuples $e_{i,j}=(c_1,\dots,c_\ell)$ such that there exist $i,j\in 1,...,l$ with $i\neq j$:

$$c_k=\begin{cases}1, & \text{if } k=i;\\ p-1, & \text{if } k=j;\\ 0, & \text{otherwise.}\end{cases}$$

Theorem 5. *[dCRdFK10]* The graph $H_{\ell,p}=(V_{\ell,p},E_{\ell,p})$ is the Cayley graph of the group $(V_{\ell,p},+)$ with the generating set $C_{\ell,p}$.

3 Diameter Graph $H_{\ell,p}$

In this section, we study the diameter of the graph $H_{\ell,p}$. The *distance* $d(u,v)$ between the vertices u and v in a graph is the number of edges in a shortest path connecting them. The *diameter* D is the largest distance among all pairs of vertices.

Lemma 3 shows that Cayley graphs are vertex transitive, thus the problem of finding the distance between two vertices can be reduced to finding the distance between a vertex v and the identity vertex. By [LJD93], $d(v)$ is defined as the length of a minimum generating sequence for the element v. So, in order to find the diameter it is sufficient to calculate the greatest distance between the identity vertex and any other vertex. The vertices that attain this distance are called *diametral*.

Observe that $d(v)=d(-v)$. In [dCRdFK12], the authors introduced some special vertices to find the diameter of the graph $H_{\ell,p}$. For the sake of clarity, we will rewrite the following definition and results, found in [dCRdFK12], using a simplified notation.

Definition 6. *[dCRdFK12](3.1)* Let $x,a\in\mathbb{Z}$, where $0\leq x<p$ and $1\leq a\leq\ell$ are such that $\ell x+(\ell-a)$ is a multiple of p. Let $x(a)=$

$(x(a)_1, x(a)_2, ..., x(a)_\ell)$ be a vertex of $H_{\ell,p}$ such that:

$$x(a)_i = \begin{cases} x, & 1 \le i \le a; \\ x+1 \mod p, & a < i \le \ell. \end{cases}$$

The distance of those vertices is easy to compute as shown in the following Lemma.

Lemma 7. *[dCRdFK12](3.5)* Consider the graph $H_{\ell,p}$ and $x(a) \in V_{\ell,p}$. There exists $b \in \mathbb{Z}$, where $0 \le b < p$, such that $(\ell - b)p = \ell x + (\ell - a)$ and

$$d(x(a)) = s_b = \begin{cases} b \cdot x, & b \le a; \\ b \cdot x + (b-a), & \text{otherwise.} \end{cases}$$

We will show that the diameter of $H_{\ell,p}$ can be deduced from $d(x(a))$.

Lemma 8. *[dCRdFK12](3.3)* Consider the graph $H_{\ell,p}$ and $v \in V_{\ell,p}$. There exist $x \in \mathbb{Z}_p$ and $a \in \mathbb{Z}$, where $0 \le a < \ell$, such that the sum of all components of x is equal to the sum of all components of $x(a)$.

$$\text{We have that } d(v) \le d(x(a)).$$

Therefore, one can obtain the diameter with the following result.

Theorem 9. *[dCRdFK12](3.1)* The diameter of the graph $H_{\ell,p}$ is

$$D = max(d(x(a))) \text{ for } x(a) \in V_{\ell,p}.$$

Moreover, the diameter of the graph $H_{\ell,p}$ equals the diameter of $H_{p,\ell}$. With this result one can assume that $\ell \le p$.

Theorem 10. The graphs $H_{\ell,p}$ and $H_{p,\ell}$ have the same diameter.

Proof (Sketch). Let $x(a) \in V_{\ell,p}$. By Lemma 7, there exists $b \in \mathbb{Z}_p$ such that $(\ell - b)p = \ell x + (\ell - a)$. We can assume that $b \le a$, otherwise consider $-x(a)$ instead of $x(a)$. Therefore, $d(x(a)) = bx$. We have that $\ell(p - x) = bp + (\ell - a)$.

If $\ell - a < p$, there exists $a' \in \mathbb{Z}$, where $1 \leq a' \leq p$, such that $\ell - a = p - a'$. Hence, $b(a') \in V_{p,\ell}$. By Lemma 7, $d(b(a')) \geq bx = d(x(a))$. By Lemma 8, $d(x(a)) \leq d(H_{p,\ell})$. Therefore, it follows from Theorem 9 that $d(H_{\ell,p}) \leq d(H_{p,\ell})$.

On the other hand, consider that $\ell - a \geq p$. By the division Theorem, there exist unique $q, r \in \mathbb{Z}$ such that $\ell(p - x) = qp + r$ and $0 \leq r < p$. Consider $a' = p - r$. If $x \neq 0$, we have that $q(a') \in V_{p,\ell}$. Since $\ell - a \geq p$, we have that $b < q$. Therefore, $d(q(a')) \geq qx > bx = d(x(a))$. By Lemma 8, $d(x(a)) \leq d(H_{p,\ell})$ for all $x(a) \in V_{\ell,p}$ with $x \neq 0$. Since, if $x = 0$, we have that $d(x(a)) = 0$, the inequality holds true. Therefore, it follows from Theorem 9 that $d(H_{\ell,p}) \leq d(H_{p,\ell})$. □

We can now easily transform the problem of finding the diameter of $H_{\ell,p}$ as a quadratic integer problem with linear condition.

Lemma 11. The diameter of the graph $H_{\ell,p}$ is obtained by solving:

$$\text{maximize } b \cdot x$$

$$\begin{array}{rcl} \text{subject to:} & & \\ b & = & (\ell \cdot (p - x) - (\ell - a))/p \\ b & \leq & a \\ x & \in & \mathbb{Z}_p \\ b & \in & \mathbb{Z}_\ell \\ \ell - a & \in & \mathbb{Z}_\ell \end{array}$$

An upper bound on the diameter is useful, since it influences the running cost of a network.

Theorem 12.
$$D(H_{\ell,p}) \leq \frac{\ell \cdot p}{4}.$$

Proof (Sketch). From the above Lemma, we have that $d(H_{\ell,p}) \leq max f(x)$ where $f(x) = bx$ and $(\ell - b)p = \ell x + (\ell - a)$. Therefore, $f(x) = \frac{\ell(p-x)-(\ell-a)}{p} x \leq$

$\frac{\ell(p-x)x}{p}$. This last function reaches its maximum at $x = p/2$, thus $f(x) \leq \ell p/4$. ☐

The following proposition gives us that the bound shown above is tight. Observe that, all results in the sequel give us a way to compute the diameter in time $O(\log(\ell + p))$.

Proposition 13. If ℓ and p are even, then $D(H_{\ell,p}) = \frac{\ell \cdot p}{4}$.

Proof (Sketch). Consider $x = p/2$ and $a = \ell$. Since p and ℓ are even, we have that $x(a) \in V_{\ell,p}$. On can easily see that $d(x(a)) = \ell p/4$. ☐

The following result also gives another exact value for the diameter, valid in the case that ℓ is a multiple of p.

Proposition 14. If p is a multiple of ℓ, then

$$D(H_{\ell,p}) = \frac{p \cdot \lfloor \ell/2 \rfloor \cdot \lceil \ell/2 \rceil}{\ell}.$$

Proof (Sketch). From the above lemma, we have that $d(H_{\ell,p}) \leq max f(x)$ where $f(x) = bx$ and $(\ell - b)p = \ell x + (\ell - a)$. Since $\ell - a \in \mathbb{Z}_\ell$, we have that $\ell - a = (\ell - b)p \mod \ell = 0$. Therefore, $f(x) = p \cdot b \cdot (\ell - b)/\ell$. This function reaches is maximum at $b = \lfloor \ell/2 \rfloor$ and thus, $f(x) \leq p \cdot \lfloor \ell/2 \rfloor \cdot \lceil \ell/2 \rceil / \ell$. Consider $x = \lceil \ell/2 \rceil \cdot p/\ell$, we have that $x(\ell) \in V_{\ell,p}$ and $d(x(a)) = \ell \cdot \lfloor p/2 \rfloor \cdot \lceil p/2 \rceil / p$. ☐

This case gives us that the upper bound is not always reached.

Corollary 15. If p is a multiple of ℓ odd, then

$$D(H_{\ell,p}) = \frac{p \cdot \ell}{4} - \frac{p}{4 \cdot \ell}.$$

We can also highlight the case where $\ell = p$.

Corollary 16.

$$D(H_{p,p}) = \lfloor p/2 \rfloor \cdot \lceil p/2 \rceil.$$

One can obtain easily some special case for small values of ℓ.

Proposition 17.

$$D(H_{2,p}) = \lfloor p/2 \rfloor$$
$$D(H_{3,p}) = \lfloor 2 \cdot p/3 \rfloor$$
$$D(H_{4,p}) = 2 \cdot \lfloor p/2 \rfloor.$$

4 Conclusion

Several authors observed that Cayley graphs provide a useful and unified framework for the design of interconnection networks for parallel computers. We analysed the $H_{\ell,p}$ family of Cayley graphs, which serves as a model for interconnection networks. We have a tight upper bound for the diameter of $H_{\ell,p}$ and calculate exact diameter for some cases. We have also shown that diameter is the same if ℓ and p are exchanged.

The $H_{\ell,p}$ graphs have degree $\ell(\ell-1)$ and respective diameter is less than or equal to $\frac{\ell \cdot p}{4}$. The diameter can be found in linear time with respect to ℓ. Since $H_{\ell,p}$ has $p^{\ell-1}$ vertices, its maximum degree is logarithmic in ℓ. The established properties such as the low diameter suggest the $H_{\ell,p}$ graph as a good topology for the design of interconnection networks.

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