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Forbidden subgraph characterization of extended star directed path graphs that are not rooted directed path graphs

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Abstract

An *asteroidal triple* in a graph is a set of three non-adjacent vertices such that for any two of them there exists a path between them that does not intersect the neighborhood of the third. An *asteroidal quadruple* is a set of four non-adjacent vertices such that any three of them is an asteroidal triple. In this paper, we study a subclass of directed path graph, the class of *extended star directed path graph* i.e directed path graph which admits a directed model with exactly one vertex of degree at least three, and give a characterization for extended star directed path graphs non rooted directed path graphs in terms of asteroidal quadruples. As byproduct, we show the family of induced forbidden subgraphs for this class.

1 Introduction

A graph is *chordal* if it contains no cycle of length at least four as an induced subgraph. A classical result [Gav74] states that a graph G is chordal if and only if there is a tree T, called *clique tree*, whose vertices are the maximal cliques of the graph and for every vertex x of G the

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maximal cliques that contain x induce a subtree in the tree which we will denote by T_x. Note that G is the intersection graph of the vertex sets of subtrees $(T_x)_{x \in V(G)}$. Clique trees are also called *models* of the graph.

Natural subclass of chordal graphs are path graphs, directed path graphs, rooted directed path graphs and extended star directed path graphs.

A graph G is a *path graph* if it admits a *path model*, i.e a clique tree T such that T_x is a subpath of T for every $x \in V(G)$. A graph G is a *directed path graph* if it admits a *directed model*, i.e a clique tree T whose edges can be directed such that T_x is a directed subpath of T for every $x \in V(G)$. A graph G is a *rooted directed path graph* if it admits a *rooted model*, i.e a directed model that can be rooted.

A graph G is an *extended star directed path graph* if it admits an *extended star directed model*, i.e a directed model which has only one vertex of degree at least three. Clearly, split graphs that are directed path graphs are star directed path graphs but there exist star directed path graphs non split graphs.

An *asteroidal triple* in a graph is a set of three non-adjacent vertices such that for any two of them there exists a path between them that does not intersect the neighborhood of the third. An *asteroidal quadruple* is a set of four non-adjacent vertices such that any three of them is an asteroidal triple.

It is certainly too difficult to characterizing rooted path graph by forbidden induced subgraphs as there are too many (families of) graphs to exclude but Cameron, Hoáng and Lévêque [CHL09] suggest that rooted directed path graphs could be characterized by forbidding some particular type of asteroidal quadruples whose vertices are linked by a special connections. In [GLT15] it has been proved that directed path graphs non rooted directed path graphs have an asteroidal quadruple. In [GT13] it has been found the family of forbidden induced subgraph for a subclass of directed path graphs.

In the present work, we will only need two types of special connections which are defined in [CHL09]. A *special connection* linking a_1 and a_2 is an induced subgraphs of one of the following forms: *Type 1:* vertices a_1, w, a_2 and edges a_1w, a_2w; *Type 2:* vertices a_1, a, b, c, d, a_2 and edges a_1a, a_1b, ab, bc, cd, bd, ac, a_2c, a_2d. It has been proved in [CHL09] that if G is a directed path graph and a_1, a_2 are two non-adjacent vertices that are linked by a special connection of Type 1 or Type 2 then for every directed model T of G, the subpath $T(a_1, a_2)$ is a directed path. As a consequence of this, they have proved that if a_1, a_2, a_3, a_4 is an asteroidal quadruple of G, a directed path graph, such that $a_1, a_2(a_3, a_4)$ are linked by a special connection of Type 1 or Type 2 then G is not a rooted directed path graph.

In this paper, we define extended star directed path graph. Hence we give a characterization for extended star directed path graphs non rooted directed path graphs in terms of asteroidal quadruples. As byproduct, we show the family of forbidden induced subgraph for this class.

The paper is organized as follows: In section 2, we present definitions, notation and give properties about directed models. Finally, in section 3, we prove the characterization of extended star directed path graphs non rooted directed path graphs and show the family of forbidden induced subgraph for this class.

2 Preliminaries

A *clique* in a graph G is a set of pairwise adjacent vertices. Let $\boldsymbol{C}(G)$ be the *set of all maximal cliques* of G and $C_x(G)$ be the set of all maximal cliques that contain the vertex x of G; when the graph is known by context we only denote by C_x.

The *neighborhood* of a vertex x is the set $N(x)$ of vertices adjacent to x and the *closed neighborhood* of x is the set $N[x] = \{x\} \cup N(x)$. A vertex is *simplicial* if its closed neighborhood is a maximal clique. Two vertices x and y are *twins* if $N[x] = N[y]$ and they are *false twins* if $N(x) = N(y)$.

Let T be a clique tree. We often use capital letters to denote the vertices of a clique tree as these vertices correspond to maximal cliques of G. In order to simplify the notation, we often write $X \in T$ instead of $X \in V(T)$, and $e \in T$ instead of $e \in E(T)$. If T' is a subtree of T, then $G_{T'}$ denotes the subgraph of G that is induced by the vertices of $\cup_{X \in V(T')} X$.

Let T be a tree. For $V' \subseteq V(T)$, let $T[V']$ be the minimum subtree of T containing V'. Then for $X, Y \in V(T)$, $T[X,Y]$ is the subpath of T between X and Y. Let $T[X,Y) = T[X,Y] \setminus Y$, $T(X,Y] = T[X,Y] \setminus X$ and $T(X,Y) = T[X,Y] \setminus \{X,Y\}$.

Let T be a clique tree of G. Let $\mathbf{D}(T)$ (or $\mathbf{D}$ for short) be the vertices of T of degree at least three. Observe that if T is not a path and H is a leaf of T then there exists $Q \in T[\mathbf{D}]$ such that $T[H,Q] \cap T[\mathbf{D}] = \{Q\}$. In this case, we say that $T[H,Q]$ is a *branch* of T incident to Q. In a clique tree T, the *label* of an edge AB of T is defined as $lab(AB) = A \cap B$. We say that $e' \in E(T)$ *dominates* $e \in E(T)$ or e is *dominated by* e' if $lab(e) \subseteq lab(e')$.

When $|V(T[H_i,Q])| > 2$ in a branch $T[H_i,Q]$, we denote by $e_i = A_iB_i$, the edge of $T[H_i,Q]$ such that A_i is the neighbor of H_i and $B_i \in T[A_i,Q]$, and by e'_i an edge dominated by e_i maximally farthest from e_i if there exists.

If G is a graph and $V' \subseteq V(G)$, then $G \setminus V'$ denotes the subgraph of G induced by $V(G) \setminus V'$. If $E' \subseteq E(G)$, then $G - E'$ denotes the subgraph of G induced by $E(G) \setminus E'$. If G, G' are two graphs, then $G + G'$ denotes the graph whose vertices are $V(G) \cup V(G')$ and edges are $E(G) \cup E(G')$. Note that if T, T' are two trees such that $|V(T) \cap V(T')| = 0$, then $T + T'$ is a forest.

We denote by $ln(T)$ the number of leaves of T. The *leafage* of a chordal graph G, denoted by $l(G)$ is a minimum integer ℓ such that G admits a model T with $ln(T) = \ell$.

We say that a tree T is an *extended star* if there is only one vertex of degree at least three in T.

Theorem 1. [GLT15] Let G be a directed path graph non rooted path graph. Then G has an asteroidal quadruple.

Lemma 2.1. *[GT13] Let G be a directed path graph minimally non rooted path graph, let T be a directed model of G with $l(G) \geq 4$, let H be a leaf of T and $T[H, Q]$ be a branch of T. 1) If $e \in T(H, Q]$ then i) it has a dominated edge outside $T[H, Q]$; ii) There are at least two vertices $x, y \in lab(e)$ such that T_x and T_y have different ends towards Q. 2) Let X, Y be vertices in $T[H, Q]$ then there are not two edges in $T[X, Y]$ with the same label.*

3 Characterization of star directed path graphs non rooted path graphs.

Theorem 2. Let G be an extended star directed path graph, minimally non rooted directed path graph, let T be an extended star directed model of G and Q be the vertex of T of degree at least three. 1) If H is a leaf of T then $G \setminus H$ is a connected graph. 2) Let e and e' be edges in two branches of T with the same label. Then the simplicial vertices of the leaves of T in those branches are linked by a special connection of Type 1 or Type 2. 3) Let H_1, H_2 be leaves of T, and $e'_1 \in T[H_2, Q]$. Then the simplicial vertices of H_1 and H_2 are linked by a special connection of Type 1 or Type 2.

Proof. 1) By way of contradiction, let $e = AB$ be an edge farthest from H such that $B \in T[A, Q]$ and $lab(e) \subset H$. By our assumption, $A \neq H$. By Lemma 2.1.1.ii and since T is a star, $e \in T(H, Q]$. Let $T' = T - T[H, A)$. Clearly $G_{T'}$ is a rooted path graph. Let T'_1 be a rooted model. By our choice of e, A is a leaf of T'_1. Let $T_1 = T'_1 + T[A, H]$. It is easy to verify that T_1 is a rooted model of G, a contradiction.

2)Let $T[H_1, Q]$ and $T[H_2, Q]$ be the branches that contain e and e' respectively. As G is a chordal graph and $ln(T) \geq 3$, G has at least three simplicial vertices. Let a_1, a_2 be simplicial vertices of G such that $N[a_i] = H_i$ for $i = 1, 2$. By minimality of G, it does not have false twins then e and e' are not incident to H_1 and H_2 (otherwise a_1 and a_2 are false twins). If one of them is incident to a leaf, for example e incident to H_1, then every vertex of $lab(e')$ has the same end towards Q, which is equal

H_1, contradicting Lemma 2.1.1.ii. Hence none of them is incident to a leaf.

Let $e \in T[H_1, Q]$ and $e' \in T[H_2, Q]$ maximizing the distance among all pair of edges with the same label. By 1) $G \setminus H_i$ for $i = 1, 2$ is a connected graph.

Let A_1, A_2 be the neighbors of H_1, H_2 respectively. Since e is not incident to H_1, $|E(T[H_1, Q])| > 1$. It follows that $A_i \neq Q$ for $i = 1, 2$. Let $e_1 = A_1B_1 \in T[H_1, Q]$, B_1 may be Q. Analogously, let $e_2 = A_2B_2 \in T[H_2, Q]$. By Lemma 2.1.1.i, there exist e'_1 and e'_2 dominated edges by e_1 and e_2 respectively. As $lab(e) = lab(e')$, and T is a directed model it follows that $e'_1, e'_2 \in T[H_1, H_2]$.

In case that e'_1 is incident to H_2, by Lemma 2.1.1.ii there is $x \in lab(e'_1)$ such that $x \in H_1 \cap H_2$. Therefore, a_1 and a_2 are linked by a special connection of Type 1. Analogously if e'_2 is incident to H_1.

Finally, suppose e'_1 and e'_2 are not incident to H_1 neither H_2 respectively. As $e'_1 \in T[A_2, Q]$, $e'_2 \in T[A_1, Q]$ and $lab(e'_2) \subset A_2$ it follows that $lab(e'_2) \subset lab(e'_1)$. Similarly, $lab(e'_1) \subset lab(e'_2)$. Hence $lab(e'_1) = lab(e'_2)$. By the choice of e and e', if e'_2 is between e and H_1 then e'_1 is between e' and Q. In this case $lab(e) \subseteq lab(e'_1) = lab(e'_2)$ and $lab(e'_1) = lab(e'_2) \subseteq lab(e)$ then $lab(e'_1) = lab(e)$. Thus $lab(e'_1) = lab(e) = lab(e') = lab(e'_2)$. As a consequence of Lemma 2.1.2, $e'_1 = e'$ and $e'_2 = e$. Therefore, every vertex in $lab(e'_1) = lab(e'_2)$ is in $A_1 \cap A_2$ and there exist $x, y \in A_1 \cap A_2$ where $x \in H_1$ and $y \in H_2$. Among all $x \in H_1 \cap A_2$, we choose one that maximizes $|C_x|$. Analogously for y. If $x \in H_2$ or $y \in H_1$ then a_1 and a_2 are linked by a special connection of Type 1.

Suppose $x \in H_1 \setminus H_2$ and $y \in H_2 \setminus H_1$. By Lemma 2.1.1.ii, there are $x_1 \neq x$ and $x_2 \neq y$ with $x_1 \in H_1 \cap A_1$ and $x_2 \in H_2 \cap A_2$ respectively. Among all the $x_1 \in H_1 \cap A_1$, we choose one minimizing $|C_{x_1}|$. Analogously for x_2. If x_1 and x_2 are adjacent then some of them would be in $lab(e'_1)$. Suppose that $x_1 \in lab(e'_1)$ then $x_1 \in lab(e_2)$ and $x_1 \in H_1 \cap A_2$. By the choice of x_1, $C_x = C_{x_1}$ so x and x_1 are twins in G, a contradiction. Therefore x_1 and x_2 are not adjacent. Thus a_1 and a_2 are linked by a

special connection of Type 2.
3) If $e'_1 = H_2A_2$, by Lemma 2.1.1.ii, there is $x \in H_1 \cap H_2$. Thus there is a special connection of Type 1. Otherwise, $e'_1 \in T(H_2, Q]$ so there exists e'_2, and it must be in $T[H_1, Q]$ due to the fact that G is a directed path graph. Then $e'_2 = H_2A_2$ or $lab(e'_1) = lab(e'_2)$. By 2), there is a special connection of Type 1 or Type 2. ■

In [CHL09] has been proved that if G is a directed path graph that has an asteroidal quadruple linked by a special connection then G is not a rooted directed path graph. The following theorem prove the converse for extended star directed path graphs.

Theorem 3. Let G be an extended star directed path graph, minimally non rooted path graph. Then $l(G) = 4$ and there is an extended star directed model that reaches the leafage. Moreover, there is an asteroidal quadruple a_1, a_2, a_3, a_4 such that a_i, a_j and a_k, a_l are linked by a special connection of Type 1 or Type 2 where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

Proof. Let T be an extended star directed model of G and Q be the vertex of degree at least three in T. By Theorem 1, $l(G) > 3$ then T has at least four leaves. Let $H_1, .., H_n$ be the leaves of T and $a_i \in H_i$ be the simplicial vertices of H_i for $i = 1, .., n$. By Theorem 2.1, $G \setminus H_i$ is a connected graph for $i = 1, .., n$. Therefore $a_1, .., a_n$ is an asteroidal n-tuple. Suppose that $n > 4$. If $|V(T)| = n + 1$ then H_iQ are the edges of T for $i = 1, .., n$. As G is non rooted path graph it follows that T can not be rooted. Thus there are two vertices x, y of G and four leaves H_1, H_2, H_3, H_4 of T such that $x \in H_1 \cap H_2$ and $y \in H_3 \cap H_4$. Clearly a_1, a_2 and a_3, a_4 are linked by a special connection of Type 1 in G. Let $G_1 = G \setminus \{a_5, .., a_n\}$. Since $a_5, .., a_n$ are simplicial vertices of G it follows that G_1 is a connected graph . As $G \setminus H_i$ is a connected graph for all $i \in \{1, .., n\}$, $G_1 \setminus H_i$ is also a connected graph for $i = 1, 2, 3, 4$. Hence in every model of G_1, H_1, H_2, H_3, H_4 are leaves. On the other hand, G_1 is a rooted path graph then there is a rooted model T'_1 of G_1. But there exist $x \in H_1 \cap H_2$ and $y \in H_3 \cap H_4$ such that $T'_{1x} = T[H_1, H_2]$ and $T'_{1y} = T[H_3, H_4]$. Hence T'_1 can not be rooted, a contradiction.

In case that $|V(T)| > n + 1$ then there is at least three vertices on $T[H_i, Q]$ for some $i \in \{1, .., n\}$. Suppose that $i = 1$. Clearly there is $e_1 \in T(H_1, Q]$ and by Lemma 2.1.1.i, e_1 has a dominated edge outside $T[H_1, Q]$. As T is a star model then there is $j \neq 1$ such that $e'_1 \in [H_j, Q]$. Suppose that $j = 2$. By Theorem 2 there is a special connection of Type 1 or 2 between a_1 and a_2. It follows that $T(a_1, a_2)$ is a directed path. Moreover as $lab(e'_1) \subset lab(e_1)$ and by Lemma 2.1.ii there is $x \in lab(e'_1) \cap H_1$.

In case that there is $i \neq 1, 2$ such that $|V(T[H_i, Q])| > 2$ then there is $e_i \in T(H_i, Q]$. By Lemma 2.1.1.i, there is $j \neq i$ such that $e'_i \in T[H_j, Q]$, and by Theorem 2, there is a special connection of Type 1 or 2 between a_i and a_j. As there is $x \in lab(e'_1) \cap H_1$ and G is a directed path graph, $e'_i \notin T[H_1, Q]$ so $j \neq 1$. Since $e'_1 \in T[H_2, Q]$ so $j \neq 2$. Suppose that $i = 3$ and $j = 4$. Let $G_1 = G \setminus \{a_5, .., a_n\}$. As $G \setminus H_i$ is a connected graph for all $i \in \{1, .., n\}$ then $G_1 \setminus H_i$ is also a connected graph for $i = 1, 2, 3, 4$. Hence in every model of G_1, H_1, H_2, H_3, H_4 are leaves. On the other hand, G_1 is a rooted path graph then there is a rooted model T'_1 of G_1. But there are special connections of Type 1 or Type 2 between a_1, a_2 and a_3, a_4 in G_1. Thus $T'_1(a_1, a_2)$ and $T'_1(a_3, a_4)$ are directed path. Thus T'_1 can not be rooted, a contradiction.

In case that for all $i \neq 1, 2$, $|V(T[H_i, Q])| = 2$, as $T(a_1, a_2)$ is a directed path and T is not a rooted model, there are $i, j \neq 1, 2$ and y a vertex of G such that $y \in H_i \cap H_j$. Suppose that $i = 3$ and $j = 4$. It follows that there is a special connection of Type 1 between a_3 and a_4 in G. Thus $T(a_3, a_4)$ is a directed path. Let $G_1 = G \setminus \{a_5, .., a_n\}$. As $G \setminus H_i$ is a connected graph for all $i \in \{1, .., n\}$ then $G_1 \setminus H_i$ is a connected graph for $i = 1, 2, 3, 4$. Hence in every model of G_1, H_1, H_2, H_3, H_4 are leaves. On the other hand, G_1 is a rooted path graph then there is a rooted model T'_1 of G_1. But there exists a special connection of Type 1 or 2 between a_1 and a_2 and a special connection of Type 1 between a_3 and a_4 in G_1. It is follows that $T'_1(a_1, a_2)$ and $T'_1(a_3, a_4)$ are directed path. Hence T'_1 can not

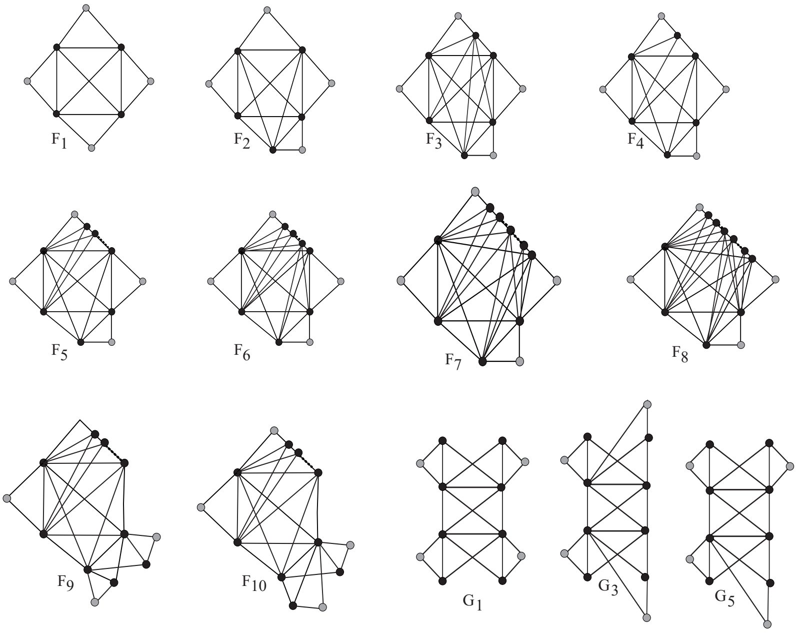

Figure 1: Family of forbidden induced subgraphs

be rooted, a contradiction. Therefore $l(G) = 4$.

■

It is easy to check that graphs shown in Figure 1 are extended star directed path graphs minimally non rooted directed path graphs. Finally, in order to build the family of forbidden induced subgraphs of the class of extended star directed path graphs non rooted directed path graphs, we take an extended star directed tree that has four leaves and we analyze all special connection of Type 1 and Type 2. Then, we obtain the following corollary.

Corollary 1. A graph is an extended star directed path graph non rooted path graph if and only if it contains one of the graphs in the families: F_1,...,F_{10}, G_1, G_3, G_5 as an induced subgraph.

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