

Carleman Inequality and Null Controllability for Parabolic Equations

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Dedicated to Professor Gervásio Colares on the occasion of his 80th birthday.

Abstract

This paper is concerned with a detailed exposition on the Carleman inequality for a parabolic equation. Specifically, it represents only a part of the work of A. V. Fursikov & O. Yu Imanovilov [7] for the particular model $p_t - \Delta p + f(p) = h$ of the heat equation. Moreover, we study the null controllability employing fixe points for multi-valued mapping.

1 Introduction

Let us consider the nonlinear parabolic state equation:

$$\left\{ \begin{array}{l} p_t(x, t) - \Delta p(x, t) + g(p(x, t)) = \chi_w u(x, t) \quad \text{in } Q, \\ p(x, t) = 0 \quad \text{on } \Sigma, \\ p(x, 0) = p_0(x) \quad \text{in } \Omega. \end{array} \right. \quad (1.1)$$

We represent by Ω a connect open set of \mathbb{R}^n with C^2 boundary $\Gamma = \partial\Omega$. For $T > 0$, real number, we consider the cylinder $Q = \Omega \times (0, T)$ of \mathbb{R}^{n+1} , with lateral boundary $\Sigma = \Gamma \times (0, T)$. The points of Ω are represented by

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$x = (x_1, \dots, x_n)$, $x_i \in \mathbb{R}$, $i = 1, \dots, n$ and those of Q are represented by (x, t) , with $x \in \Omega$ and $0 < t < T$. By w we consider a subset of Ω , that is, $w \subset \Omega$. The real functions $p = p(x, t)$, $u = u(x, t)$ defined on Q are the state and the control respectively. All the derivatives are in the sense of the theory of distributions of Laurent-Schwartz. By p_t we represent the partial derivative $\partial p / \partial t$ and Δ is the Laplace operator, that is, $\partial^2 / \partial x_1^2 + \partial^2 / \partial x_2^2 + \dots + \partial^2 / \partial x_n^2$. With $p_0(x)$ we denote the initial data of the initial boundary value problem (1.1): χ_w is the characteristic function of w .

The function $g: \mathbb{R} \rightarrow \mathbb{R}$, is $C^1(\mathbb{R})$, globally Lipschitz, that is

$$|g(p_1) - g(p_2)| \leq M|p_1 - p_2| \text{ for all } p_1, p_2 \in \mathbb{R} \text{ and } g(0) = 0.$$

LINEARIZED SYSTEM

We define $f: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(p) = \begin{cases} \frac{g(p)}{p} & \text{if } |p| > 0, \\ g'(0) & \text{if } p = 0. \end{cases}$$

We define, employing the function f , a linearized system associated with (1.1) given by

$$\begin{cases} p_t(x, t) - \Delta p(x, t) + f(\bar{p}(x, t))p(x, t) = \chi_w u(x, t) & \text{in } Q, \\ p(x, t) = 0 & \text{on } \Sigma, \\ p(x, 0) = p_0(x) & \text{in } \Omega. \end{cases} \quad (1.2)$$

DEFINITIONS

(i) The system (1.2) is said to be approximately controllable in $L^2(\Omega)$, at time $T > 0$, if for each $\varepsilon > 0$, given $p_0 \in L^2(\Omega)$ and $p_T(x) \in L^2(\Omega)$, there exists a control $u \in L^2(Q_w)$, $Q_w = w \times (0, T)$, such that the corresponding solution $p(x, t)$ of (1.2) satisfies

$$|p(x, T) - p_T(x)|_{L^2(\Omega)} < \varepsilon.$$

By $L^2(\Omega)$ we represent the Lebesgue space of square integrable functions on Ω with the inner product and norm:

$$(u, v)_{L^2(\Omega)} = \int_{\Omega} u(x)v(x) dx \quad \text{and} \quad |u|_{L^2(\Omega)}^2 = \int_{\Omega} u(x)^2 dx,$$

where u and v are real valued functions ■

(ii) The system (1.2) is said to be null controllable at time $T > 0$, if for each $p_0 \in L^2(\Omega)$, there exists a control $u \in L^2(Q_w)$ such that the solution p of (1.2) satisfies $p(x, T) = 0$ a.e. in Ω ■

We consider a real function $a(x, t)$ uniformly bounded in the sense

$$|a(x, t)|_{L^\infty(Q)} < M.$$

In the sequel $a(x, t) = f(\bar{p}(x, t))$. Thus, we are concerned, initially, with the adjoint system of (1.2) which is given by

$$\begin{cases} w_t(x, t) + \Delta w(x, t) - a(x, t)w(x, t) = f_1(x, t) & \text{in } Q, \\ w(x, t) = 0 & \text{on } \Sigma, \\ w(x, T) = w_T & \text{in } \Omega, \end{cases} \quad (1.3)$$

with $w_T \in L^2(\Omega)$ and $f_1 \in L^2(Q)$.

In the next section we prove the **Carleman inequality** for the adjoint system (1.3), following the method of Fursikov-Imanovilov [6]. In this methodology it is fundamental the following result:

Lemma 1.1: *Let $w_0 \subset w \subset \Omega$ a nonempty open subset. Then, there exists a function $\psi \in C^2(\bar{\Omega})$, $\bar{\Omega}$ closure of Ω , such that*

$$\begin{cases} \psi(x) > 0 \text{ for all } x \in \Omega, \\ \psi = 0 \text{ for all } x \in \Gamma, \\ |\nabla \psi(x)| > 0 \text{ for } x \in \Omega - w_0. \end{cases}$$

The proof of this Lemma can be found in Fursikov-Imanovilov [7]. From Lemma 1.1 we introduce the weight functions

$$\phi(x, t) = \frac{e^{\lambda\psi(x)}}{\beta(t)} \quad \text{and} \quad \alpha(x, t) = \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|}}{\beta(t)}, \quad (1.4)$$

with $\beta(t) = t(T - t)$, $0 < t < T$, $\lambda > 0$ a real parameter and

$$\|\psi\| = \max_{x \in \bar{\Omega}} |\psi(x)|.$$

From (1.4) we verify that

$$\nabla \phi = \lambda \frac{e^{\lambda\psi}}{\beta(t)} \nabla \psi = \lambda \phi \nabla \psi = \nabla \alpha \quad \blacksquare \quad (1.5)$$

2 Carleman inequality

All this paragraph is dedicated to prove the inequality of Carleman for solution w of the adjoint system (1.3). In the method of Fursikov-Imanovilov [7] is crucial the results of Lemma 1.1. The main result is contained in the following theorem.

Theorem 2.1. *Let ψ, ϕ, α be the functions defined above. Then, there exist positive constants λ_0, s_0 and C such that*

$$\int_Q \left[(s\phi)^{-1} (|w_t|^2 + |\Delta w|^2) + \lambda^2 s\phi |\nabla w|^2 + \lambda^4 (s\phi)^3 |w|^2 \right] e^{2s\alpha} dxdt \leq \\ C \int_Q e^{2s\alpha} |f_1|^2 dxdt + C \int_{Q_w} e^{2s\alpha} \lambda^4 (s\phi)^3 |w|^2 dxdt,$$

for all $s \geq s_0$ and $\lambda \geq \lambda_0$, where $s_0 = s_1(\Omega, \omega)(T + T^2)$, $\lambda_0 = \lambda_0(\Omega, \omega)$, $C = C(\Omega, \omega)$, $w = w(x, t)$ is solution of the adjoint system (1.3), $|\cdot|$ is the absolute value of real numbers and s_1 is a suitable constant.

Remark 2.1. *Setting $p(x, t) = (s\phi)^\ell e^{s\alpha(x, t)} w(x, t)$, we get*

$$\int_Q \left[(s\phi)^{\ell-1} (|w_t|^2 + |\Delta w|^2) + \lambda^2 (s\phi)^{\ell+1} |\nabla w|^2 + \lambda^4 (s\phi)^{\ell+3} |w|^2 \right] e^{2s\alpha} dxdt \leq \\ C \int_Q (s\phi)^\ell e^{2s\alpha} |f_1|^2 dxdt + C \int_{Q_w} e^{2s\alpha} \lambda^4 (s\phi)^{\ell+3} |w|^2 dxdt,$$

for all $\ell \in \mathbb{Z}$. A look at the proof of Theorem 2.1 shows that the proof of this remark can be carried out in exactly the same way.

The above inequality is called **Carleman Inequality**. The proof of Theorem 2.1 is very much technical. It will be done by steps, following Fursikov-Imanovilov [7].

Step 1. We consider a convenient change of variables to introduce in the adjoint system (1.3) by the regularization function, that is, $e^{s\lambda\alpha(x, t)}$. In fact, setting

$$w(x, t) = e^{-s\alpha(x, t)} p(x, t) \quad \text{or} \quad p(x, t) = e^{s\alpha(x, t)} w(x, t),$$

we obtain

$$w_t(x, t) = -s\alpha_t e^{-s\alpha} p + e^{-s\alpha} p_t. \tag{2.1}$$

Besides that,

$$\begin{aligned}\frac{\partial w}{\partial x_i} &= -s \frac{\partial \alpha}{\partial x_i} e^{-s\alpha} p + e^{-s\alpha} \frac{\partial p}{\partial x_i}, \\ \frac{\partial^2 w}{\partial x_i^2} &= -s \frac{\partial^2 \alpha}{\partial x_i^2} e^{-s\alpha} p + s^2 \left(\frac{\partial \alpha}{\partial x_i} \right)^2 e^{-s\alpha} p - 2s \frac{\partial \alpha}{\partial x_i} e^{-s\alpha} \frac{\partial p}{\partial x_i} + e^{-s\alpha} \frac{\partial^2 p}{\partial x_i^2}.\end{aligned}$$

Thus, we obtain

$$\Delta w = -s \Delta \alpha e^{-s\alpha} p + s^2 |\nabla \alpha|^2 e^{-s\alpha} p - 2s e^{-s\alpha} \nabla \alpha \cdot \nabla p + e^{-s\alpha} \Delta p.$$

From (1.5) we also have

$$\Delta \alpha = \nabla \cdot (\nabla \alpha) = \lambda \nabla \phi \cdot \nabla \psi + \lambda \phi \Delta \psi = \lambda^2 \phi |\nabla \psi|^2 + \lambda \phi \Delta \psi,$$

and $|\nabla \alpha|^2 = \lambda^2 \phi^2 |\nabla \psi|^2$. Therefore, we find

$$\begin{aligned}\Delta w &= -s(\lambda^2 \phi |\nabla \psi|^2 + \lambda \phi \Delta \psi) e^{-s\alpha} p + \\ & s^2 \lambda^2 \phi^2 |\nabla \psi|^2 e^{-s\alpha} p - 2s \lambda \phi e^{-s\alpha} \nabla \psi \cdot \nabla p + e^{-s\alpha} \Delta p.\end{aligned}\tag{2.2}$$

From (2.1) and (2.2) and the system (1.3) we obtain

$$\begin{aligned}e^{-s\alpha} p_t - e^{-s\alpha} (s \alpha_t p) + e^{-s\alpha} (-2s \phi \nabla \phi \cdot \nabla p + s^2 \lambda^2 \phi^2 |\nabla \psi|^2 p - \\ s \lambda^2 \phi |\nabla \psi|^2 p + \Delta p - s \lambda \phi \Delta \psi p) = f_1 + a(t) e^{-s\alpha} p.\end{aligned}\tag{2.3}$$

We also have that

$$p(x, 0) = e^{s\alpha(x,0)} w(x, 0) = 0 \quad \text{in } \Omega,\tag{2.4}$$

because

$$\alpha(x, t) = \frac{e^{\lambda \psi(x)} - e^{2\lambda \|\psi\|}}{\beta(t)} < 0 \quad \text{and} \quad e^{s\alpha(x,0)} = \lim_{t \rightarrow 0^+} e^{s\alpha(x,t)} = 0.$$

By similar argument, we obtain

$$p(x, T) = e^{s\alpha(x,T)} w(x, T) = 0 \quad \text{in } \Omega.$$

Then, from (2.3) and (2.4) we re-write the state equation (1.3), in the new variables, given by

$$\left\{ \begin{array}{l} p_t - \alpha_t s p - 2s \lambda \phi \nabla \psi \cdot \nabla p + s^2 \lambda^2 \phi^2 |\nabla \psi|^2 p - \\ \quad s \lambda^2 \phi |\nabla \psi|^2 p + \Delta p - s \lambda \phi \Delta \psi p = e^{s\alpha} f_1 + a(t) p \quad \text{in } Q, \\ p(x, t) = 0 \quad \text{on } \Sigma, \\ p(x, 0) = p(x, T) = 0 \quad \text{in } \Omega. \end{array} \right.\tag{2.5}$$

Let us consider the following notation.

$$\begin{cases} U(t)p = -2s\lambda^2\phi|\nabla\psi|^2 p - 2s\lambda\phi\nabla\psi \cdot \nabla p, \\ V(t)p = -\Delta p - s^2\lambda^2\phi^2|\nabla\psi|^2 p - s\lambda^2\phi|\nabla\psi|^2 p + \alpha_t s p, \\ Z(t)p = s\lambda\phi\Delta\psi p + a(t)p. \end{cases} \tag{2.6}$$

With the notation (2.6) we re-write the equation (2.5)₁, as follows

$$p_t + U(t)p - V(t)p = e^{s\alpha} f_1 + Z(t)p. \tag{2.7}$$

Note that

$$\frac{d}{dt} \int_{\Omega} (V(t)p)p \, dx = \int_{\Omega} (V(t)p_t)p \, dx + \int_{\Omega} (V(t)p)p_t \, dx + \int_{\Omega} (V_t(t)p)p \, dx.$$

The two first integrals can be written as $2 \int_{\Omega} (V(t)p)p_t \, dx$. Substituting p_t , which is given by (2.7), we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} (V(t)p)p \, dx = \\ & 2 \int_{\Omega} (V(t)p)(e^{s\alpha} f_1 + Z(t)p - U(t)p + V(t)p) \, dx + \int_{\Omega} (V_t(t)p)p \, dx. \end{aligned} \tag{2.8}$$

Integrating (2.8) from 0 to T and observing that $p(0) = p(x, 0)$ and $p(T) = p(x, T)$ are zero on Ω , we obtain

$$\begin{aligned} 0 = & 2 \int_Q (V(t)p)^2 \, dxdt + 2 \int_Q (V(t)p)(e^{s\alpha} f_1 + Z(t)p) \, dxdt + \\ & \int_Q (V_t(t)p)p \, dxdt + 2 \left(- \int_Q (V(t)p)(U(t)p) \, dxdt \right). \end{aligned} \tag{2.9}$$

Analysis of the terms of (2.9). Denoting by X the last integral of (2.9), we have

$$\begin{aligned} X = & - \int_Q (V(t)p)(U(t)p) \, dxdt = \\ & = - \int_Q (\Delta p + s^2\lambda^2\phi^2|\nabla\psi|^2 p + s\lambda^2\phi|\nabla\psi|^2 p - \alpha_t s p) \\ & \quad \cdot (2s\lambda^2\phi|\nabla\psi|^2 p + 2s\lambda\nabla\psi \cdot \nabla p) \, dxdt. \end{aligned} \tag{2.10}$$

Remark 2.2. From the definition of ϕ and α , we obtain

$$|\phi_t| = \left| \frac{\beta'(t)}{\beta^2(t)} e^{\lambda\psi(x)} \right| = \frac{|T - 2t|}{e^{\lambda\psi(x)}} |\phi|^2 \leq C \phi^2, \tag{2.11}$$

where the constant C depends on T , λ , $\|\psi\|$ and Ω .

$$|\alpha_t| = \left| -\frac{\beta'(t)}{\beta^2(t)} (e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|}) \right| \leq \frac{|T - 2t| |e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|}|}{e^{2\lambda\psi(x)}} \frac{e^{2\lambda\psi(x)}}{\beta^2(t)} \leq C \phi^2. \tag{2.12}$$

$$\begin{aligned} |\alpha_{tt}| &= \left| \frac{-\beta''\beta^2 + 2\beta|T - 2t|^2}{\beta^4} \right| |e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|}| \leq \\ &\leq \left| \frac{2\beta^2 + 2\beta|T - 2t|^2}{\beta(t)e^{3\lambda\psi(x)}} \right| \left(\frac{e^{3\lambda\psi(x)}}{\beta^3} \right) |e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|}| = \\ &= \left| \frac{2\beta + 2|T - 2t|^2}{e^{2\lambda\psi(x)}} \right| \frac{|e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|}|}{e^{\lambda\psi(x)}} \phi^3 \leq C \phi^3. \end{aligned} \tag{2.13}$$

Note, the constant depends on λ , T and Ω .

$$\left| \alpha \frac{d}{dt} \ln \beta^{-1}(t) \right| = \left| \alpha \frac{\beta'(t)}{\beta(t)} \right| = \left| (T - 2t) \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|}}{e^{2\lambda\psi(x)}} \right| \frac{e^{2\lambda\psi(x)}}{\beta^2(t)} \leq C \phi^2 \quad \blacksquare \tag{2.14}$$

From (2.12)–(2.14) of Remark 2.2 we return to (2.9) and set

$$\begin{aligned} X_1 &= \left| \int_Q (V_t(t)p)p \, dxdt \right| = \\ &\left| \int_0^T \int_\Omega (-\Delta p_t - s^2\lambda^2(2\phi\phi_t)|\nabla\psi|^2 p - s\lambda^2\phi_t|\nabla\psi|^2 p + \alpha_{tt} sp)p \, dxdt \right| \leq \\ &\frac{1}{2} \int_0^T \frac{d}{dt} |\nabla p|_{L^2(\Omega)}^2 dt + C_1 \int_Q (\lambda^2 s^2 \phi^3 + s\lambda^2 \phi^2 + s\phi^3) |p|^2 \, dxdt, \end{aligned} \tag{2.15}$$

where C_1 depends on Ω and T . Note that the integral of the derivative of $|\nabla p|_{L^2(\Omega)}^2$ is zero. We also get from (2.9) that

$$\begin{aligned} X_2 &= \left| \int_Q 2(V(t)p)(e^{s\alpha} f_1 + Z(t)p) \, dxdt \right| \leq \\ &\int_Q |2(V(t)p)e^{s\alpha} f_1| \, dxdt + \int_Q |2(V(t)p)Z(t)p| \, dxdt \leq \\ &2 \int_Q |V(t)p|^2 \, dxdt + \int_Q e^{2s\alpha} |f_1|^2 \, dxdt + \int_Q |Z(t)p|^2 \, dxdt. \end{aligned} \tag{2.16}$$

From the definition of $Z(t)p$, we obtain

$$\begin{aligned} \int_Q |Z(t)p|^2 \, dxdt &= \int_Q |s\lambda\phi\Delta\psi p + a(t)p|^2 \, dxdt \\ &\leq C \int_Q (s^2\lambda^2\phi^2 + M) |p|^2 \, dxdt. \end{aligned} \tag{2.17}$$

Thus, from (2.9), (2.15), (2.16) and (2.17), we have

$$\begin{aligned} & 2X + 2 \int_Q |V(t)p|^2 dxdt \leq \\ & C \int_Q (\lambda^2 s^2 \phi^3 + s\lambda^2 \phi^2 + s\phi^3) |p|^2 dxdt + \\ & 2 \int_Q |V(t)p|^2 dxdt + \int_Q e^{2s\alpha} |f_1|^2 dxdt + C \int_Q (s^2 \lambda^2 \phi^2 + M) |p|^2 dxdt. \end{aligned}$$

Then, we obtain

$$X \leq C \int_Q [\lambda^2 (s^2 \phi^2 + s^2 \phi^3) + s\phi^3 + 1] |p|^2 dxdt + \frac{1}{2} \int_Q e^{2s\alpha} |f_1|^2 dxdt. \quad (2.18)$$

Step 2. In this step we calculate X by another process. In fact, we have

$$X = - \int_Q (V(t)p)(U(t)p) dxdt.$$

Otherwise

$$\begin{aligned} X &= - \int_Q (\Delta p + s^2 \lambda^2 \phi^2 |\nabla \psi|^2 p + \\ & s\lambda^2 \phi |\nabla \psi|^2 - \alpha_t s p) (2s\lambda^2 \phi |\nabla \psi|^2 p + 2s\lambda \phi \nabla \psi \cdot \nabla p) dxdt = \\ & 2 \int_Q (s\lambda^2 \phi |\nabla \psi|^2 p) \Delta p dxdt - 2 \int_Q s^3 \lambda^4 \phi^3 |\nabla \psi|^4 p^2 dxdt - \\ & 2 \int_Q s^2 \lambda^4 \phi^2 |\nabla \psi|^4 p^2 dxdt - 2 \int_Q (s\lambda \phi \nabla \psi \cdot \nabla p) \Delta p dxdt - \\ & 2 \int_Q (s^3 \lambda^3 \phi^3 |\nabla \psi|^2 \nabla \psi \cdot \nabla p p + s^2 \lambda^3 \phi^2 |\nabla \psi|^2 \nabla \psi \cdot \nabla p p) dxdt + \\ & 2 \int_Q (s^2 \lambda^2 \phi \alpha_t |\nabla \psi|^2 p + s^2 \lambda \phi \alpha_t \nabla \psi \cdot \nabla p) p dxdt. \end{aligned} \quad (2.19)$$

Now we employ the notation

$$\begin{aligned} M_1 &= -2 \int_Q (s\lambda^2 \phi |\nabla \psi|^2 p) \Delta p dxdt; \\ M_2 &= -2 \int_Q s^3 \lambda^4 \phi^3 |\nabla \psi|^4 p^2 dxdt - 2 \int_Q s^2 \lambda^4 \phi^2 |\nabla \psi|^4 p^2 dxdt; \\ M_3 &= -2 \int_Q (s\lambda \phi \nabla \psi \cdot \nabla p) \Delta p dxdt; \\ M_4 &= -2 \int_Q (s^3 \lambda^3 \phi^3 |\nabla \psi|^2 \nabla \psi \cdot \nabla p p + s^2 \lambda^3 \phi^2 |\nabla \psi|^2 \nabla \psi \cdot \nabla p p) dxdt; \\ M_5 &= 2 \left(\int_Q (s^2 \lambda^2 \phi \alpha_t |\nabla \psi|^2 p + s^2 \lambda \phi \alpha_t \nabla \psi \cdot \nabla p) p dxdt. \right. \end{aligned}$$

Thus,

$$X = M_1 + M_2 + M_3 + M_4 + M_5.$$

The next steps are to calculate the integrals M_i for $i = 1, \dots, 5$. In fact, applying Green's formula to M_1 and observing that $p = 0$ on Σ , we obtain

$$\begin{aligned} M_1 &= -2 \int_Q ((s\lambda^2\phi)|\nabla\psi|^2 p)\Delta p \, dxdt = & (2.20) \\ & 2 \int_Q s\lambda^2 \nabla(\phi|\nabla\psi|^2 p) \cdot \nabla p \, dxdt = 2 \int_Q s\lambda^3 \phi |\nabla\psi|^2 p \nabla\psi \cdot \nabla p \, dxdt + \\ & 2 \int_Q s\lambda^2 \phi \nabla(|\nabla\psi|^2) \cdot \nabla p p \, dxdt + 2 \int_Q s\lambda^2 \phi |\nabla\psi|^2 |\nabla p|^2 \, dxdt. \end{aligned}$$

As $\phi(x, t) = e^{\lambda\psi(x)}/\beta(t)$, then $\nabla\phi = \lambda\phi\nabla\psi$. From this and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} \left| 2 \int_Q s\lambda^3 \phi |\nabla\psi|^2 p \nabla\psi \cdot \nabla p \, dxdt \right| &\leq & (2.21) \\ 4 \int_Q s\lambda^4 \phi |\nabla\psi|^4 p^2 \, dxdt + \frac{1}{4} \int_Q s\lambda^2 \phi |\nabla\psi|^2 |\nabla p|^2 \, dxdt. \end{aligned}$$

$$\begin{aligned} \left| 2 \int_Q s\lambda^2 \phi \nabla(|\nabla\psi|^2) \cdot \nabla p p \, dxdt \right| &= & (2.22) \\ \left| 2 \int_Q s\lambda^2 \phi 2|\nabla\psi|(\nabla|\nabla\psi|) \cdot \nabla p p \, dxdt \right| &\leq \\ 16 \int_Q s\phi\lambda^2 |\nabla(|\nabla\psi|)|^2 p^2 \, dxdt + \frac{1}{4} \int_Q s\lambda^2 \phi |\nabla\psi|^2 |\nabla p|^2 \, dxdt. \end{aligned}$$

Remark 2.3. As $\phi(x, t) = e^{\lambda\psi(x)}/\beta(t)$ with $\psi \in C^2(\overline{\Omega})$, then we have $|\nabla\psi|^4 < C$ and $|\nabla(|\nabla\psi|)|^2 < C$, where C is a positive constant depending only of Ω .

Thus, from Remark 2.3, (2.21) and (2.22) we transform (2.20) as follows.

$$M_1 \geq \frac{3}{2} \int_Q s\lambda^2 \phi |\nabla\psi|^2 |\nabla p|^2 \, dxdt - C \int_Q s\phi(\lambda^4 + \lambda^2)|p|^2 \, dxdt. \quad (2.23)$$

Applying Green's formula in M_3 , we get

$$M_3 = 2 \int_Q s\lambda \nabla(\phi \nabla\psi \cdot \nabla p) \cdot \nabla p \, dxdt - 2 \int_{\Sigma} (s\lambda\phi \nabla\psi \cdot \nabla p) \nabla p \cdot n \, d\Sigma.$$

Observe that $\nabla p \cdot n = \frac{\partial p}{\partial n}$, where n is the exterior unit vector normal to Σ . After same calculus, it implies

$$\begin{aligned} M_3 &= 2 \int_Q s\lambda^2 \phi (\nabla\psi \cdot \nabla p)^2 \, dxdt + 2 \int_Q s\lambda\phi \psi_{x_i x_j} p_{x_i} p_{x_j} \, dxdt + \\ & 2 \int_Q s\lambda\phi \psi_{x_i} p_{x_i x_j} p_{x_j} \, dxdt - 2 \int_{\Sigma} (s\lambda\phi \nabla\psi \cdot \nabla p) \nabla p \cdot n \, d\Sigma. \end{aligned}$$

Setting

$$N_1 = 2 \int_Q s \lambda \phi \psi_{x_i} p_{x_i x_j} p_{x_j} dx dt,$$

and observing that $p_{x_i x_j} p_{x_j} = \frac{1}{2} \left((p_{x_j})^2 \right)_{x_i}$, we obtain

$$N_1 = 2 \int_Q s \lambda \phi \psi_{x_i} \frac{1}{2} \left((p_{x_j})^2 \right)_{x_i} dx dt.$$

Auxiliary Computations. By Gauss Lemma, we obtain

$$\int_Q \frac{\partial}{\partial x_i} \left(s \lambda \phi \psi_{x_i} (p_{x_j})^2 \right) dx dt = \int_{\Sigma} s \lambda \phi \psi_{x_i} (p_{x_j})^2 n_i d\Sigma.$$

Applying the derivative $\frac{\partial}{\partial x_i}$, we find

$$\begin{aligned} & \int_Q s \lambda \phi \psi_{x_i} \frac{\partial}{\partial x_i} \left(\frac{\partial p}{\partial x_j} \right)^2 dx dt + \int_Q s \lambda \phi_{x_i} \psi_{x_i} (p_{x_j})^2 dx dt + \\ & \int_Q s \lambda \phi \psi_{x_i x_i} (p_{x_j})^2 dx dt = \int_{\Sigma} s \lambda \phi \psi_{x_i} (p_{x_j})^2 n_i d\Sigma. \end{aligned}$$

Inserting the above equality in the integral N_1 , it yields

$$\begin{aligned} N_1 &= - \int_Q s \lambda \phi_{x_i} \psi_{x_i} (p_{x_j})^2 dx dt - \int_Q s \lambda \phi \psi_{x_i x_i} (p_{x_j})^2 dx dt \\ &+ \int_{\Sigma} s \lambda \phi (p_{x_j})^2 \psi_{x_i} n_i d\Sigma. \end{aligned}$$

Since $\phi(x, t) = e^{\lambda\psi(x)}/\beta(t)$ then $\phi_{x_i} = \lambda\phi\psi_{x_i}$. Thus, $s\lambda\phi_{x_i}\psi_{x_i} = s\lambda\phi(\psi_{x_i})^2$.

Therefore, we obtain

$$\begin{aligned} N_1 &= - \int_Q s \lambda^2 \phi (\psi_{x_i})^2 (p_{x_j})^2 dx dt - \int_Q s \lambda \phi \psi_{x_i x_i} (p_{x_j})^2 dx dt \\ &+ \int_{\Sigma} s \lambda \phi (p_{x_j})^2 \psi_{x_i} n_i d\Sigma. \end{aligned}$$

Thus,

$$\begin{aligned} N_1 &= - \int_Q s \lambda^2 \phi |\nabla \psi|^2 |\nabla p|^2 dx dt - \int_Q s \lambda \phi \Delta \psi |\nabla p|^2 dx dt \\ &+ \int_{\Sigma} s \lambda \phi \frac{\partial \psi}{\partial n} |\nabla p|^2 d\Sigma. \end{aligned}$$

Inserting this identity in M_3 , we obtain

$$\begin{aligned} M_3 = & 2 \int_Q s \lambda^2 \phi (\nabla \psi \cdot \nabla p)^2 dxdt + 2 \int_Q s \lambda \phi \psi_{x_i x_j} p_{x_i} p_{x_j} dxdt - \\ & 2 \int_\Sigma s \lambda \phi (\nabla \psi \cdot \nabla p) \nabla p \cdot n d\Sigma - \int_Q s \lambda^2 \phi |\nabla \psi|^2 |\nabla p|^2 dxdt - \\ & \int_Q s \phi \Delta \psi |\nabla p|^2 dxdt + \int_\Sigma s \lambda \phi \frac{\partial \psi}{\partial n} |\nabla p|^2 d\Sigma. \end{aligned} \quad (2.24)$$

Remark 2.4. *The two surface integrals of (2.24) satisfy*

$$\int_0^T \int_\Gamma s \lambda \phi \frac{\partial \psi}{\partial n} |\nabla p|^2 d\Gamma dt \leq 0 \quad \text{and} \quad -2 \int_0^T \int_\Gamma s \lambda \phi (\nabla \psi \cdot \nabla p) \nabla p \cdot n d\Sigma \geq 0.$$

In fact, since ψ satisfies the conditions of Lemma 1.1, that is, $\psi \in C^2(\bar{\Omega})$; $\psi = 0$ on $\Gamma = \partial\Omega$ and $\psi > 0$ in Ω , thus, by definition

$$\frac{\partial \psi}{\partial n}(x) = \lim_{k \rightarrow 0^-} \frac{\psi(x + kn) - \psi(x)}{k} = \lim_{k \rightarrow 0^-} \frac{\psi(x + kn)}{k} < 0,$$

because $\psi(x) = 0$ on Γ and $\psi(x) > 0$ for $x \in \Omega$. If $x \in \Gamma$, $k < 0$, then $x + kn \in \Omega$, where n exterior unit normal to Γ . Thus, $\frac{\partial \psi}{\partial n} < 0$ on Γ and

$$\int_\Sigma s \lambda \phi \frac{\partial \psi}{\partial n} |\nabla p|^2 d\Sigma \leq 0. \quad (2.25)$$

For the second surface integral, $\nabla \psi = n \frac{\partial \psi}{\partial n}$ and $\frac{\partial \psi}{\partial n} < 0$ on Γ , then

$$-2 \int_\Sigma s \phi \lambda \left(\frac{\partial \psi}{\partial n} n \cdot \nabla p \right) \frac{\partial p}{\partial n} d\Sigma = \int_\Sigma s \phi \lambda \frac{\partial \psi}{\partial n} \left(\frac{\partial p}{\partial n} \right)^2 d\Sigma \geq 0 \quad \blacksquare \quad (2.26)$$

Thus, using (2.25) in (2.24), we obtain

$$\begin{aligned} M_3 - N_2 \leq & 2 \int_Q s \lambda^2 \phi (\nabla \psi \cdot \nabla p)^2 dxdt - \\ & \int_Q s \lambda^2 \phi |\nabla \psi|^2 |\nabla p|^2 dxdt + 2 \int_Q s \lambda \phi \psi_{x_i x_j} p_{x_i} p_{x_j} dxdt - \int_Q s \phi \Delta \psi |\nabla p|^2 dxdt, \end{aligned} \quad (2.27)$$

where $N_2 = -2 \int_\Sigma s \lambda \phi \nabla \psi \cdot \nabla p \nabla p \cdot n d\Sigma$.

$$|M_3 - N_2| \leq \int_Q s \lambda^2 \phi |\nabla \psi|^2 |\nabla p|^2 dxdt + C \int_Q s \lambda \phi |\nabla p|^2 dxdt, \quad (2.28)$$

where we have used that $|\Delta\psi|$ and $|\psi_{x_i x_j}|$ are bounded in Ω , due to hypótese on ψ of Lemma 1.1. Thus, from (2.28), we obtain

$$M_3 - N_2 \geq - \int_Q s\lambda^2 |\nabla\psi|^2 |\nabla p|^2 dxdt - C \int_Q s\lambda\phi |\nabla p|^2 dxdt$$

and consequently

$$M_3 \geq - \int_Q s\lambda^2 \phi |\nabla\psi|^2 |\nabla p|^2 dxdt - C \int_Q s\lambda\phi |\nabla p|^2 dxdt. \quad (2.29)$$

Thus, from (2.23) and (2.29), we have

$$\begin{aligned} M_1 + M_3 &\geq \frac{1}{2} \int_Q s\lambda^2 \phi |\nabla\psi|^2 |\nabla p|^2 dxdt - \\ &C \int_Q s\phi(\lambda^4 + \lambda^2) |\nabla p|^2 dxdt - C \int_Q s\lambda\phi |\nabla p|^2 dxdt. \end{aligned} \quad (2.30)$$

where $C > 0$ is a constant that depends on Ω .

Now, we are working with M_4 . In fact,

$$\begin{aligned} M_4 &= -2 \int_Q (s^3 \lambda^3 \phi^3 |\nabla\psi|^2 \nabla\psi \cdot \nabla p p + s^2 \lambda^3 \phi^2 |\nabla\psi|^2 \nabla\psi \cdot \nabla p p) dxdt = \\ &- \int_Q (s^3 \lambda^3 \phi^3 |\nabla\psi|^2 \nabla\psi \nabla p^2 + s^2 \lambda^3 \phi^2 |\nabla\psi|^2 \nabla\psi \cdot \nabla p^2) dxdt = \\ &\int_Q (s^3 \lambda^3 \nabla \cdot (\phi^3 |\nabla\psi|^2 \nabla\psi) |p|^2 + s^2 \lambda^3 \nabla \cdot (\phi^2 |\nabla\psi|^2 \nabla\psi) |p|^2) dxdt = \\ &3 \int_Q (s^3 \lambda^4 \phi^3 |\nabla\psi|^4 |p|^2) dxdt + \int_Q s^3 \lambda^3 \phi^3 \nabla \cdot (|\nabla\psi|^2 \nabla\psi) |p|^2 dxdt + \\ &2 \int_Q s^2 \lambda^4 \phi^2 |\nabla\psi|^4 |p|^2 dxdt + \int_Q s^2 \lambda^3 \phi^2 \nabla \cdot (|\nabla\psi|^2 \nabla\psi) |p|^2 dxdt. \end{aligned} \quad (2.31)$$

Note that $\left| \nabla \left(|\nabla\psi|^2 \nabla\psi \right) \right| < C$ and $\nabla\phi^3 = 3\phi^2 \nabla\phi = 3\phi^2 (\lambda\phi \nabla\psi) = 3\lambda\phi^3 \nabla\psi$. Thus, $\lambda\phi^3 \nabla\psi = \frac{1}{3} \nabla\phi^3$. Also we have $\nabla\phi^2 = 2\phi \nabla\phi = 2\lambda\phi^2 \nabla\psi$. Therefore, we modify (2.31) to find

$$\begin{aligned} M_4 &\geq \int_Q \lambda^4 (3s^3 \phi^3 + 2s^2 \phi^2) |\nabla\psi|^4 |p|^2 dxdt \\ &- C \int_Q (\lambda^3 s^3 \phi^3 + \lambda^3 s^2 \phi^2) |p|^2 dxdt. \end{aligned} \quad (2.32)$$

Thus, from (2.32) and the definition of M_2 , we obtain

$$\begin{aligned} M_2 + M_4 &\geq \int_Q s^3 \lambda^4 \phi^3 |\nabla\psi|^4 |p|^2 dxdt \\ &- C \int_Q (\lambda^3 s^3 \phi^3 + \lambda^2 s^2 \phi^2) |p|^2 dxdt. \end{aligned} \quad (2.33)$$

Finally, we will modify M_4 . From (2.11) and (2.14), we obtain

$$M_5 = 2 \int_Q s^2 \lambda^2 \phi \alpha_t |\nabla \psi|^2 |p|^2 dxdt + \int_Q s^2 \lambda \phi \alpha_t \nabla \psi \cdot \nabla (p^2) dxdt = \\ 2 \int_Q s^2 \lambda^2 \phi \alpha_t |\nabla \psi|^2 |p|^2 dxdt - \int_Q s^2 \lambda \nabla (\phi \alpha_t \nabla \psi) |p|^2 dxdt.$$

Applying Gauss Lemma in the second integral and as $p = 0$ on Σ , we get

$$M_5 = 2 \int_Q s^2 \lambda^2 \phi \alpha_t |\nabla \psi|^2 |p|^2 dxdt - \int_Q s^2 \lambda^2 \phi \alpha_t |\nabla \psi|^2 |p|^2 dxdt - \\ \int_Q s^2 \lambda \phi \alpha_t (\lambda \phi_t |\nabla \psi|^2) |p|^2 dxdt - \int_Q s^2 \lambda \phi \alpha_t \Delta \psi |p|^2 dxdt.$$

Hence, we obtain

$$|M_5| \leq C \left(\int_Q s^2 \lambda^2 \phi^3 |\nabla \psi|^2 |p|^2 dxdt + \int_Q s^2 \lambda \phi^3 |\Delta \psi| |p|^2 dxdt \right).$$

Thus,

$$M_5 \geq -C \int_Q (s^2 \lambda^2 + s^2 \lambda) \phi^3 p^2 dxdt. \tag{2.34}$$

From (2.18), (2.19), (2.30), (2.33) and (2.34) we have

$$\frac{1}{2} \int_Q s \lambda^2 \phi |\nabla \psi|^2 |\nabla p|^2 dxdt + \int_Q s^3 \lambda^4 \phi^3 |\nabla \psi|^4 |p|^2 dxdt \leq \tag{2.35} \\ C \int_Q s \phi (\lambda^4 + \lambda^2) |p|^2 dxdt + C \int_Q s \lambda \phi |\nabla p|^2 dxdt + \\ C \int_Q (\lambda^3 s^3 \phi^3 + \lambda^3 s^2 \phi^2) |p|^2 dxdt + C \int_Q (s^2 \lambda^2 + s^2 \lambda) \phi^3 |p|^2 dxdt + \\ C \int_Q (\lambda^2 (s^2 \phi^2 + s^2 \phi^3) + s \phi^3 + 1) |p|^2 dxdt + \int_Q e^{2s\alpha} |f_1|^2 dxdt \leq \\ C \int_Q (s \lambda \phi |\nabla p|^2 + e^{2s\alpha} |f_1|^2) dxdt + \int_Q \phi^3 (s^2 \lambda^4 + s^3 \lambda^3 + 1) |p|^2 dxdt.$$

Since $\lambda \geq \lambda_0 \geq 1$, $s \geq s_0(\lambda) \geq 1$ and $C < |\phi|$, we have

$$s \phi (\lambda^4 + \lambda^2) |p|^2 \leq \lambda^4 s^2 \phi^3 |p|^2 \quad \text{and} \quad \lambda^3 s^3 \phi^2 p^2 \leq \lambda^3 s^3 \phi^3 |p|^2.$$

Besides, since $|\nabla \psi| > 0$ in $\partial\Omega$, then $|\nabla \psi| > 0$ in the compact set $\partial\Omega \cup (\Omega - \omega_0)$.

Therefore, there exists $\gamma > 0$ such that $0 < \gamma < |\nabla \psi|$ for all $x \in \partial\Omega \cup (\Omega - \omega_0)$ and for all $x \in \Omega - \omega_0$. About the open sets ω and ω_0 look at the Lemma 1.1.

With this in mind we get from (2.35), that

$$\int_{Q-Q\omega_0} (\gamma^4 s^3 \lambda^4 \phi^3 p^2 + \gamma^2 s \lambda^2 \phi |\nabla p|^2) dxdt \leq \tag{2.36} \\ C \left(\int_Q e^{2s\alpha} |f_1|^2 dxdt + \int_Q (s \lambda \phi |\nabla p|^2 + (s^3 \lambda^3 + s^2 \lambda^4 + 1) \phi^3 p^2) dxdt \right).$$

Hence,

$$\int_{Q-Q_{\omega_0}} (s^3\lambda^4\phi^3p^2 + s\lambda^2\phi|\nabla p|^2) dxdt \leq \tag{2.37}$$

$$C\left(\int_Q e^{2s\alpha}|f_1|^2 dxdt + \int_Q [s\lambda\phi|\nabla p|^2 + (s^3\lambda^3 + s^2\lambda^4 + 1)\phi^3p^2] dxdt\right).$$

For $\lambda > \lambda_0$ and $s \geq s_0(\lambda)$ sufficiently large we get from (2.37) that

$$\frac{1}{2} \int_{Q-Q_{\omega_0}} (s^3\lambda^4\phi^3p^2 + s\lambda^2\phi|\nabla p|^2) dxdt \leq \tag{2.38}$$

$$C\left(\int_Q e^{2s\alpha}|f_1|^2 dxdt + \int_{Q_{\omega_0}} (s\lambda\phi|\nabla p|^2 + s^3\lambda^3\phi^3p^2) dxdt\right),$$

for s such that $s^2\lambda^4 \leq s^3\lambda^3$ (for example $\lambda = s$). Therefore,

$$\int_Q (s^3\lambda^4\phi^3p^2 + s\lambda^2\phi|\nabla p|^2) dxdt \leq \int_{Q_{\omega_0}} (s^3\lambda^4\phi^3p^2 + s\lambda^2\phi|\nabla p|^2) dxdt + \tag{2.39}$$

$$2C\left(\int_Q e^{2s\alpha}|f_1|^2 dxdt + \int_{Q_{\omega_0}} (s\phi\lambda|\nabla p|^2 + s^3\lambda^3\phi^3p^2) dxdt\right) \leq$$

$$(1 + 2C)\left(\int_Q e^{2s\alpha}|f_1|^2 dxdt + \int_{Q_{\omega_0}} (s^3\lambda^3\phi^3p^2 + s\lambda\phi|\nabla p|^2)\right) dxdt \leq$$

$$C_1\left(\int_Q e^{2s\alpha}|f_1|^2 dxdt + \int_{Q_{\omega_0}} (s^3\lambda^4\phi^3p^2 + s\lambda^2\phi|\nabla p|^2) dxdt\right).$$

Therefore, we have

$$\int_Q (s^3\lambda^4\phi^3p^2 + s\lambda^2\phi|\nabla p|^2) dxdt \leq \tag{2.40}$$

$$C_1\left(\int_Q e^{2s\alpha}|f_1|^2 dxdt + \int_{Q_{\omega_0}} (s^3\lambda^4\phi^3p^2 + s\lambda^2\phi|\nabla p|^2) dxdt\right).$$

Step 3. (Return to original variables)

As $p = e^{s\alpha}w$, then $\nabla p = s \nabla\alpha e^{s\alpha}w + e^{s\alpha}\nabla w$, and it implies $|\nabla p|^2 \leq 2s^2e^{2s\alpha}|\nabla\alpha|^2|w|^2 + 2e^{2s\alpha}|\nabla w|^2$ and $|\nabla\alpha|^2 \leq \lambda^2\phi^2|\nabla\psi|^2$. Thus, we obtain

$$|\nabla p|^2 \leq C e^{2s\alpha}(s^2\lambda^2\phi^2|w|^2 + |\nabla w|^2). \tag{2.41}$$

Besides, we also have

$$\nabla p = s \nabla\alpha p + e^{s\alpha} \nabla w, \tag{2.42}$$

and so

$$|\nabla p|^2 = s^2|\nabla\alpha|^2 p^2 + e^{2s\alpha} |\nabla w|^2 + 2se^{s\alpha}(\nabla\alpha \cdot \nabla w)p.$$

Using (2.41) and (2.42) in (2.40), we have

$$\begin{aligned} & \int_Q [e^{2s\alpha} \lambda^4 s^3 \phi^3 |w|^2 + s\lambda^2 \phi (s^2 |\nabla \alpha|^2 p^2 + \\ & e^{2s\alpha} |\nabla w|^2 + 2se^{s\alpha} (\nabla \alpha \cdot \nabla w)p)] dxdt \leq C \left[\int_{Q_{\omega_0}} (s\lambda^2 \phi C e^{2s\alpha} (s^2 \lambda^2 \phi^2 |w|^2 + |\nabla w|^2) + \right. \\ & \left. e^{2s\alpha} s^3 \lambda^4 \phi^3 |w|^2) dxdt + \int_Q e^{2s\alpha} |f_1|^2 dxdt \right] \leq \\ & C \left[\int_{Q_{\omega_0}} (e^{2s\alpha} s^3 \lambda^4 \phi^3 |w|^2 + e^{2s\alpha} s\lambda^2 \phi |\nabla w|^2) dxdt + \int_Q e^{2s\alpha} |f_1|^2 dxdt \right]. \end{aligned}$$

Re-write this inequality in a convenient form, we have

$$\begin{aligned} & \int_Q e^{2s\alpha} s^3 \lambda^4 \phi^3 |w|^2 dxdt + \int_Q s^3 \lambda^2 \phi |\nabla \alpha|^2 p^2 dxdt + \quad (2.43) \\ & \int_Q s\lambda^2 \phi e^{2s\alpha} |\nabla w|^2 dxdt + 2 \int_Q s^2 \lambda^2 \phi e^{s\alpha} (\nabla \alpha \cdot \nabla w)p dxdt \leq \\ & C \int_{Q_{\omega_0}} s^3 \lambda^4 \phi^3 e^{2s\alpha} |w|^2 dxdt + C \int_{Q_{\omega_0}} s\lambda^2 \phi e^{2s\alpha} |\nabla w|^2 dxdt \\ & + C \int_Q e^{2s\alpha} |f_1|^2 dxdt. \end{aligned}$$

From (2.43) if

$$\begin{aligned} N_3 &= C \int_{Q_{\omega_0}} s^3 \lambda^4 \phi^3 e^{2s\alpha} |w|^2 dxdt + C \int_{Q_{\omega_0}} s\lambda^2 \phi e^{2s\alpha} |\nabla w|^2 dxdt \\ &+ \int_Q e^{2s\alpha} |f_1|^2 dxdt, \end{aligned}$$

then we can see directly

$$\int_Q s^3 \lambda^2 \phi |\nabla \alpha|^2 p^2 dxdt \leq N_3. \quad (2.44)$$

Moreover,

$$\begin{aligned} & \left| 2 \int_Q s^2 \lambda^2 \phi e^{s\alpha} |\nabla \alpha \cdot \nabla w| p dxdt \right| \leq \quad (2.45) \\ & 2 \int_Q s^3 \lambda^2 \phi |\nabla \alpha|^2 p^2 dxdt + \frac{1}{2} \int_Q s\lambda^2 \phi e^{2s\alpha} |\nabla w|^2 dxdt. \end{aligned}$$

Thus, from (2.43), (2.44), (2.45), we obtain

$$\begin{aligned} & \int_Q (s^3 \lambda^4 \phi^3 e^{2s\alpha} |w|^2 + s\lambda^2 \phi e^{2s\alpha} |\nabla w|^2) dxdt \leq \quad (2.46) \\ & N_3 + N_3 + 2N_3 + \frac{1}{2} \int_Q s\lambda^2 \phi e^{2s\alpha} |\nabla w|^2 dxdt. \end{aligned}$$

Hence,

$$\int_Q (s^3 \lambda^4 \phi^3 e^{2s\alpha} |w|^2 + s \lambda^2 \phi e^{2s\alpha} |\nabla w|^2) dxdt \leq \tag{2.47}$$

$$C \left[\int_{Q_{w_0}} (s^3 \lambda^4 \phi^3 e^{2s\alpha} |w|^2 + s \lambda^2 \phi e^{2s\alpha} |\nabla w|^2) dxdt + \int_Q e^{2s\alpha} |f_1|^2 dxdt \right] \blacksquare$$

Step 4. (Carleman Estimates: Conclusion of the proof of Theorem 2.1)

Let us consider the square of both sides of the state equation (1.3)₁. After, multiply both sides by $(s\phi)^{-1} e^{2s\alpha}$ and integrate on Q , we obtain

$$\int_Q e^{2s\alpha} (s\phi)^{-1} (|w_t|^2 + |\Delta w|^2) dxdt = \tag{2.48}$$

$$\int_Q e^{2s\alpha} (s\phi)^{-1} |f_1|^2 dxdt + \int_Q e^{2s\alpha} (s\phi)^{-1} |w|^2 |a(t)|^2 dxdt +$$

$$2 \int_Q e^{2s\alpha} (s\phi)^{-1} f_1 a(t) w dxdt - 2 \int_Q e^{2s\alpha} (s\phi)^{-1} w_t \Delta w dxdt.$$

Now we will examine each term of (2.48). First, note that

$$(s\phi)^{-1} \leq C \quad \text{and} \quad (s\phi)^{-1} \leq C(s\phi)^3. \tag{2.49}$$

From this and (2.49), we obtain

$$\left| \int_Q e^{2s\alpha} (s\phi)^{-1} |f_1|^2 dxdt \right| \leq C \int_Q e^{2s\alpha} |f_1|^2 dxdt; \tag{2.50}$$

$$\left| \int_Q (s\phi)^{-1} e^{2s\alpha} |w|^2 |a(t)|^2 dxdt \right| \leq MC_1 \int_Q e^{2s\alpha} s^3 \phi^3 |w|^2 dxdt. \tag{2.51}$$

As $\alpha(0) = \alpha(T) = -\infty$, then $e^{2s\alpha(0)} = e^{2s\alpha(T)} = 0$. Thus, by Gauss Lemma,

we get

$$\begin{aligned}
& 2 \int_Q e^{2s\alpha} (s\phi)^{-1} w_t \Delta w \, dxdt = \tag{2.52} \\
& - 2 \int_Q \nabla(e^{2s\alpha} (s\phi)^{-1} w_t) \cdot \nabla w \, dxdt = \\
& - 2 \int_Q 2s \nabla \alpha e^{2s\alpha} (s\phi)^{-1} w_t \nabla w \, dxdt + \\
& + 2 \int_Q e^{2s\alpha} s^{-1} \phi^{-2} \nabla \phi \cdot \nabla w w_t \, dxdt - \int_Q e^{2s\alpha} (s\phi)^{-1} \frac{d}{dt} |\nabla w|^2 \, dxdt = \\
& - 4 \int_Q \lambda e^{2s\alpha} w_t \nabla \psi \cdot \nabla w \, dxdt + 2 \int_Q e^{2s\alpha} (s\phi)^{-1} \lambda w_t \nabla \psi \cdot \nabla w \, dxdt + \\
& \int_Q (2s\alpha_t e^{2s\alpha} (s\phi)^{-1} - e^{2s\alpha} s^{-1} \phi^{-2} \phi_t) |\nabla w|^2 \, dxdt \leq \\
& C \int_Q e^{2s\alpha} |\nabla \psi| |\nabla w| |w_t| \, dxdt + C \int_Q e^{2s\alpha} (s\phi) |\nabla w|^2 \, dxdt \leq \quad (C < s\phi) \\
& C \int_Q e^{2s\alpha} (s\phi) |\nabla w|^2 \, dxdt + \frac{1}{2} \int_Q e^{2s\alpha} (s\phi)^{-1} |w_t|^2 \, dxdt.
\end{aligned}$$

Substituting (2.49), (2.50), (2.51) and (2.52) in (2.48), we obtain

$$\begin{aligned}
& \int_Q e^{2s\alpha} (s\phi)^{-1} \left(\frac{1}{2} |w_t|^2 + |\Delta w|^2 \right) \, dxdt \leq \tag{2.53} \\
& C \left[\int_Q e^{2s\alpha} |f_1|^2 \, dxdt + \int_Q e^{2s\alpha} s^3 \phi^3 |w|^2 \, dxdt + \int_Q e^{2s\alpha} s\phi |\nabla w|^2 \, dxdt \right] \leq \\
& C_1 \left[\int_Q e^{2s\alpha} |f_1|^2 \, dxdt + \int_{Q_{\omega_0}} e^{2s\alpha} s^3 \phi^3 |w|^2 \, dxdt + \right. \\
& \left. \int_{Q_{\omega_0}} e^{2s\alpha} s\phi |\nabla w|^2 \, dxdt \right].
\end{aligned}$$

Note that Q_{ω_0} and $\frac{1}{2} |w_t|^2$ comes from (2.47) and (2.52), (2.48) respectively.

Let us consider a function $\chi \in C_0^\infty(\Omega)$ such that $\chi = 1$ in $\bar{\omega}_0$, closure of ω_0 , and $\chi = 0$ on $\Omega - \omega$. Multiplying the adjoint system (1.3) by $e^{2s\alpha} \chi s\phi w$ and integrating on Q , we obtain

$$\begin{aligned}
& \int_Q e^{2s\alpha} \chi s\phi w w_t \, dxdt + \int_Q e^{2s\alpha} \chi s\phi w \Delta w \, dxdt - \tag{2.54} \\
& \int_Q e^{2s\alpha} \chi s\phi a(t) w^2 \, dxdt = \int_Q e^{2s\alpha} \chi s\phi f_1 \, dxdt.
\end{aligned}$$

Analysis of the terms of (2.54). We have

$$\begin{aligned} \int_Q e^{2s\alpha} \chi s \phi w w_t dxdt &= \frac{1}{2} \int_{Q_\omega} (e^{2s\alpha} \chi s \phi) \frac{d}{dt} w^2 dxdt = \\ &- \frac{1}{2} \int_{Q_\omega} (e^{2s\alpha} s \phi)_t |w|^2 dxdt \leq \frac{1}{2} \int_{Q_\omega} \chi (2s\alpha_t e^{2s\alpha} s \phi + e^{2s\alpha} \phi_t) w^2 dxdt \leq \\ &C \int_{Q_\omega} e^{2s\alpha} s^3 \phi^3 w^2 dxdt, \quad \text{for } s \geq s_1 > 1. \end{aligned} \quad (2.55)$$

Applying the Green's formula and observing that $w = 0$ on Σ , we get

$$\begin{aligned} \int_Q e^{2s\alpha} \chi s \phi |\nabla w|^2 dxdt &= \\ - \int_Q e^{2s\alpha} \chi s \phi w \Delta w dxdt &- \int_Q s w \nabla (e^{2s\alpha} \chi \phi) \nabla w. \end{aligned} \quad (2.56)$$

On the other hand,

$$\begin{aligned} \left| \int_Q s w \nabla (e^{2s\alpha} \chi \phi) \cdot \nabla w dxdt \right| &= \\ \left| \int_{Q_\omega} s w \left[\chi (2s \nabla \alpha e^{2s\alpha} \phi + e^{2s\alpha} \alpha \phi \nabla \psi) + e^{2s\alpha} \phi \nabla \chi \right] (\nabla w) dxdt \right| &\leq \\ C \varepsilon \int_{Q_\omega} s \phi e^{2s\alpha} w^2 dxdt + \frac{1}{2\varepsilon} \int_{Q_\omega} e^{2s\alpha} \chi s \phi |\nabla w|^2 dxdt + \\ \frac{1}{2C_1} \int_Q e^{2s\alpha} (s\phi)^{-1} |\Delta w|^2 dxdt. \end{aligned} \quad (2.57)$$

Now adding (2.47) with (2.53), we get

$$\begin{aligned} \int_Q e^{2s\alpha} (s\phi)^{-1} \left(\frac{1}{2} |w_t|^2 + |\Delta w|^2 \right) dxdt &+ \\ \int_Q (e^{2s\alpha} s^3 \lambda^4 \phi^3 |w|^2 + e^{2s\alpha} s \lambda^2 \phi e^{2s\alpha} |\nabla w|^2) dxdt &\leq \\ C \left[\int_{Q_{\omega_0}} e^{2s\alpha} s^3 \lambda^4 \phi^3 |w|^2 dxdt + \right. \\ \left. \int_{Q_{\omega_0}} e^{2s\alpha} s \lambda^2 \phi |\nabla w|^2 dxdt + \int_Q e^{2s\alpha} |f_1|^2 dxdt \right]. \end{aligned} \quad (2.58)$$

From (2.56), we obtain

$$\begin{aligned}
\int_{Q_{\omega_0}} e^{2s\alpha} s\lambda^2 \phi |\nabla w|^2 dxdt &\leq \int_Q e^{2s\alpha} \chi s\lambda^2 \phi |\nabla w|^2 dxdt \leq \\
\left| \int_Q e^{2s\alpha} \chi (s\phi) w \Delta w dxdt \right| + C\varepsilon \int_{Q_{\omega_0}} e^{2s\alpha} (s\phi)^3 w^2 dxdt + & \quad (2.59) \\
\frac{1}{\varepsilon} \int_{Q_{\omega_0}} e^{2s\alpha} (s\phi) |\nabla w|^2 dxdt &\leq \\
\frac{1}{2C_1} \int_{Q_\omega} e^{2s\alpha} (s\phi)^{-1} |\Delta w|^2 dxdt + C\varepsilon \int_{Q_\omega} e^{2s\alpha} s^3 \lambda^4 \phi^3 |w|^2 dxdt + & \\
\frac{1}{\varepsilon} \int_{Q_\omega} e^{2s\alpha} s\lambda^2 \phi |\nabla w|^2 dxdt. &
\end{aligned}$$

Finally, from (2.58), (2.59) and choosing $\varepsilon > 0$, such that $C_2(\lambda)/\varepsilon = 1/2$ (to bring the term in $|\nabla w|^2$ to the left-hand side), we get

$$\begin{aligned}
\int_Q \left[(s\phi)^{-1} (|w_t|^2 + |\Delta w|^2) + s^3 \lambda^4 \phi^3 |w|^2 + s\lambda^2 \phi |\nabla w|^2 \right] e^{2s\alpha} dxdt &\leq \\
C \left[\int_{Q_\omega} e^{2s\alpha} s^2 \lambda^4 \phi^2 |w|^2 dxdt + \int_Q e^{2s\alpha} |f_1|^2 dxdt \right], &
\end{aligned}$$

where $w = w(x, t)$ is weak solutions of the adjoint system (1.3). It was proved by Fursikov-Imanovilov [7], see also Fursikov [6].

3 Observability inequality

In the present section we will prove the observability inequality for weak solutions of the adjoint system (1.3). Observe that it is a consequence of the Carleman Inequality proved in Section 2.

Theorem 3.1. *Suppose α, ϕ as in Theorem 2.1. Then, for $\lambda > \lambda_0 > 0$ and $s > s(\lambda) > 1$, we have*

$$\int_\Omega |w(x, 0)|^2 dx \leq C \int_Q e^{2s\alpha} |f_1|^2 dxdt + \int_{Q_\omega} e^{2s\alpha} \phi^3 |w|^2 dxdt, \quad (3.1)$$

where C is a positive constant that depends only of Ω and T .

This inequality is called *Observability Inequality* for the adjoint system (1.3).

Proof. Multiply both sides of (1.3)₁ by w and integrate on Ω , we obtain

$$-\frac{1}{2} \frac{d}{dt} \int_\Omega w^2 dxdt + \int_\Omega |\nabla w|^2 dx = \int_\Omega w f_1 dxdt - \int_\Omega a(x, t) w^2 dx. \quad (3.2)$$

Multiplying both sides of (3.2) by $e^{2(M+1)t}$ and observing that $|a(t)|_{L^\infty(\Omega)} < M$ a. e. in $[0, T[$, we obtain

$$-\frac{d}{dt} \left(e^{2(M+1)t} \int_{\Omega} w^2 dx dt \right) \leq \frac{e^{2(M+1)t}}{2} \int_{\Omega} f_1^2 dx. \tag{3.3}$$

Integrating (3.3) from 0 to t , we get:

$$|w(x, 0)|_{L^2(\Omega)}^2 \leq e^{2(M+1)t} |w(t)|_{L^2(\Omega)}^2 + \int_0^t \left(\frac{e^{2(M+1)y}}{2} \int_{\Omega} f_1(x, y)^2 dx \right) dy. \tag{3.4}$$

Setting

$$\theta(t) = \sup \{ e^{-2s\alpha(x,t)}; x \in \Omega \},$$

we have

$$\theta(t) \leq e^{(2s e^{2\lambda\|\psi\|})/\beta(t)}, \tag{3.5}$$

since $-\alpha \leq \frac{e^{2\lambda\|\psi\|}}{\beta(t)}$. Then $e^{-2s\alpha(x,t)} \leq \theta(t)$ and

$$1 \leq \theta(t) e^{2s\alpha(x,t)}. \tag{i}$$

We know that $\phi(x, t) = e^{\lambda\psi(x)}/\beta(t)$, $\beta(t) = t(T - t)$, $0 < t < T$, for $x \in \Omega$. The function ψ as in Lemma 1.1. Thus, $\frac{1}{\phi^3} \leq C$, for $0 \leq t \leq T$, $x \in \bar{\Omega}$, or

$$1 \leq C \phi^3(x, t), \quad 0 \leq t \leq T, \quad x \in \bar{\Omega}. \tag{ii}$$

If $1 \leq s^3$, then by (3.5), (i) and (ii) we obtain from (3.4), that

$$\begin{aligned} |w(x, 0)|_{L^2(\Omega)}^2 &\leq C_T \int_{\Omega} |w(x, t)|^2 dx + C_T \int_0^t \left(\frac{1}{2} \int_{\Omega} |f_1(x, y)|^2 dx \right) dy \\ &\leq C \theta(t) \int_{\Omega} e^{2s\alpha} |w(x, t)|^2 dx + C \theta(t) \int_0^t \left(\int_{\Omega} e^{2s\alpha(x,t)} |f_1(x, y)|^2 dx \right) dy. \end{aligned}$$

Thus,

$$\frac{1}{\theta(t)} |w(x, 0)|^2 \leq C \int_{\Omega} e^{2s\alpha} |w(x, t)|^2 dx + C \int_0^t \left(\int_{\Omega} e^{2s\alpha} |f_1(x, y)|^2 dx \right) dy. \tag{3.6}$$

From (3.5), we obtain

$$0 < k_T < e^{-\frac{2s e^{2\lambda\|\psi\|}}{\beta(t)}} \leq \frac{1}{\theta(t)}, \quad 0 < t < T.$$

Now, we fixe $t_1 < t_2$ in $(0, T)$ and integrate (3.6) on (t_1, t_2) with respect to t , to get

$$\begin{aligned}
 & |w(x, 0)|_{L^2(\Omega)}^2 \int_{t_1}^{t_2} e^{-\frac{2s e^{2\lambda} \|\psi\|}{\beta(t)}} dt \leq \\
 & C \int_{t_1}^{t_2} \left(\int_{\Omega} e^{2s\alpha(x,t)} |w(x, t)|^2 dx \right) dt + \\
 & C \int_{t_1}^{t_2} \left[\int_0^t \left(\int_{\Omega} e^{2s\alpha(x,t)} |f_1(x, y)|^2 dx \right) dy \right] dt.
 \end{aligned} \tag{3.7}$$

In the Appendix, established to follow, we prove

$$\int_{\Omega} \int_0^T \int_0^t e^{2s\alpha} |f_1(x, y)|^2 dx dy dt \leq C \int_0^T \int_{\Omega} |f_1(x, y)|^2 dx dy.$$

Since

$$\inf_{t \in [0, T]} e^{-\frac{2s e^{2\lambda} \|\psi\|}{\beta(t)}} \geq C_T,$$

from (3.7) and from the Appendix above cited, we obtain

$$\begin{aligned}
 & |w(x, 0)|_{L^2(\Omega)}^2 \leq C \int_0^T \int_{\Omega} e^{2s\alpha} |w(x, t)|^2 dx dt + \\
 & C \int_0^T \int_{\Omega} |f_1(x, t)|^2 dx dt.
 \end{aligned} \tag{3.8}$$

By (ii), $1 \leq C \phi^3(x, t)$, for $x \in \bar{\Omega}$ and $0 < t < T$. Then, we have from (3.8):

$$|w(x, 0)|_{L^2(\Omega)}^2 \leq C \int_0^T \int_{\Omega} e^{2s\alpha} \phi^3 |w(x, t)|^2 dx dt + C \int_0^T \int_{\Omega} |f_1(x, y)|^2 dx dy. \tag{3.9}$$

By Carleman inequality we obtain:

$$\begin{aligned}
 & \int_Q e^{2s\alpha} s^3 \phi^3 |w(x, t)|^2 dx dt \leq \\
 & C \int_Q e^{2s\alpha} |f_1(x, y)|^2 dx dy + \int_{Q_\omega} e^{2s\alpha} \phi^3 |w(x, t)|^2 dx dt.
 \end{aligned}$$

Then,

$$\begin{aligned}
 & \int_Q e^{2s\alpha} \phi^3 |w(x, t)|^2 dx dt \leq C \int_{Q_\omega} e^{2s\alpha} \phi^3 |w(x, t)|^2 dx dt + \\
 & C \int_Q e^{2s\alpha} |f_1(x, t)|^2 dx dt.
 \end{aligned}$$

Substituting the last inequality in (3.9) we obtain the **Observability Inequality** for the weak solution of the adjoint state (1.3). Note that $C > 0$ is a constant that depends on Ω and T ■

4 Null controllability: Linear State Equation

We consider the linear state equation:

$$\begin{cases} p_t(x, t) - \Delta p(x, t) + a(x, t)p(x, t) = \chi_\omega u(x, t) & \text{in } Q, \\ p(x, t) = 0 & \text{on } \Sigma, \\ p(x, 0) = p_0(x) & \text{in } \Omega. \end{cases} \quad (4.1)$$

When we have $\chi_\omega u \in L^2(Q)$, $p_0 \in L^2(\Omega)$, the weak solution p of (4.1) has the following regularity, see Brezis [3],

$$p \in H^1(0, T; H^{-1}(\Omega)) \cap L^2(0, T; H_0^1(\Omega)).$$

Thus, $p \in C^0([0, T]; L^2(\Omega))$. The null controllability for (4.1) consist in to obtain a control $u \in L^2(Q)$, such that

$$p(x, T) = 0 \quad \text{a. e. in } \Omega.$$

Observe that $a(t)$ is bounded in Q and $|a|_{L^\infty(Q)} < M$, as in Section 1.

We have the following regularity

$$\begin{aligned} p &\in W^1(0, T; H_0^1(\Omega), H^{-1}(\Omega)) \\ &= \{p \in L^2(0, T; H_0^1(\Omega), p_t \in L^2(0, T; H^{-1}(\Omega))\} \subset C^0([0, T]; L^2(\Omega)). \end{aligned}$$

Theorem 4.1. *For $p_0 \in L^2(\Omega)$, there exists a control $u \in L^2(Q)$ such that the weak solution $p = p(x, t)$ of the state equation (4.1) satisfies $p(x, T) = 0$ in Ω .*

Proof. The proof of Theorem 4.1 is done by a variational method and an application of the observability inequality, cf. Section 3. The control u picked up in $L^2(Q, e^{-2s\alpha} \phi^{-3})$ satisfies the inequality.

$$\int_{Q_\omega} e^{-2s\alpha} \phi^{-3} u^2 dxdt \leq C \int_{\Omega} p_0^2 dx. \quad (4.2)$$

Note that $e^{-2s\alpha} \phi^{-3} \geq C_0$. For each $\varepsilon > 0$, we define the functional

$$N_\varepsilon(p, u) = \int_Q e^{-2s\alpha} \phi^{-3} u^2 dxdt + \frac{1}{\varepsilon} \int_{\Omega} p(x, T)^2 dx, \quad (4.3)$$

for $u \in L^2(Q, e^{-2s\alpha} \phi^{-3})$ and p is the weak solution of (4.1), with $p_0 \in L^2(\Omega)$. Observe that $N_\varepsilon(p, u)$ is lower semi-continuous, strictly convex and coercive in $L^2(Q)$. Then, the variational problem

$$\min N_\varepsilon(p, u),$$

has a unique solution $u_\varepsilon \in L^2(Q)$.

We suppose that $u_\varepsilon \in L^2(Q)$ is the minimizer of $N_\varepsilon(p, u)$. Thus, by mean of the state equation (4.1) we find the weak solution p_ε . The next step consists to prove the convergence

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon = u \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} p_\varepsilon = p.$$

Then, we have to prove that p is the weak solution of (4.1) corresponding to the control u and that

$$p(x, T) = 0 \quad \text{a.e. in } \Omega.$$

Lemma 4.1. *We have that*

$$u_\varepsilon = e^{2s\alpha} \phi^3 \chi_\omega w_\varepsilon \quad \text{a. e. in } Q,$$

with

$$w_\varepsilon \in H^1(0, T; H^{-1}(\Omega)) \cap L^2(0, T; H_0^1(\Omega)),$$

is the weak solution of the parabolic problem

$$\left\{ \begin{array}{l} w_{\varepsilon t} + \Delta w_\varepsilon - a(t)w_\varepsilon = 0 \quad \text{in } Q, \\ w_\varepsilon = 0 \quad \text{on } \Sigma, \\ w_\varepsilon(x, T) = -\frac{1}{\varepsilon} p(x, T) \quad \text{in } \Omega, \end{array} \right. \quad (4.4)$$

being $p(x, t)$ the weak solution of (4.1), that is

$$\left\{ \begin{array}{l} p_t - \Delta p + a(t)p = \chi_\omega u_\varepsilon \quad \text{in } Q, \\ p = 0 \quad \text{on } \Sigma, \\ p(x, 0) = p_0(x) \quad \text{in } \Omega, \end{array} \right. \quad (4.5)$$

$$p_0 \in L^2(\Omega), \quad u \in L^2(Q).$$

Proof. We write $p = \widehat{p} + \bar{p}$ with \widehat{p} and \bar{p} weak solutions of the systems

$$\begin{cases} \widehat{p}_t - \Delta \widehat{p} + a(t)\widehat{p} = 0 & \text{in } Q, \\ \widehat{p} = 0 & \text{on } \Sigma, \\ \widehat{p}(x, 0) = \widehat{p}(x) & \text{in } \Omega, \end{cases} \tag{4.6}$$

and

$$\begin{cases} \bar{p}_t - \Delta \bar{p} + a(t)\bar{p} = \chi_\omega u & \text{in } Q, \\ \bar{p} = 0 & \text{on } \Sigma, \\ \bar{p}(x, 0) = 0 & \text{in } \Omega. \end{cases} \tag{4.7}$$

We observe that in (4.7) we have a linear dependence of the solution \bar{p} from the control u , which we denote by:

$$Lu = \bar{p}(x, T). \tag{4.8}$$

Note that $L: L^2(Q) \rightarrow L^2(\Omega)$ is linear and bounded, because p belongs to $C^0([0, T]; L^2(\Omega))$. Thus, we re-write $N_\varepsilon(p, u) = J_\varepsilon(u)$, with

$$J_\varepsilon(u) = \int_Q e^{-2s\alpha} \phi^{-3} u^2 dxdt + \frac{1}{\varepsilon} \int_\Omega (\widehat{p}(T) + Lu)^2 dx. \tag{4.9}$$

The stationary value $u_\varepsilon \in L^2(Q)$, for the functional $J_\varepsilon(u)$, defined by (4.9), is that in which the Gateaux derivative is null in all direction $\tilde{\omega} \in L^2(Q)$. It means,

$$J'_\varepsilon(u_\varepsilon) \cdot \tilde{\omega} = 0 \quad \text{for all } \tilde{\omega} \in L^2(Q),$$

that is,

$$\left. \frac{d}{d\lambda} J_\varepsilon(u_\varepsilon + \lambda\tilde{\omega}) \right|_{\lambda=0} = 0 \quad \text{for all } \tilde{\omega} \in L^2(Q).$$

Now, we will do the computation with the weight $e^{-2s\alpha} \phi^{-3}$ in the functional J_ε observing that $p(T) = \widehat{p}(T) + \bar{p}(T) = \widehat{p}(T) + Lu$. In fact, we obtain

$$\left. \frac{d}{d\lambda} J_\varepsilon(u_\varepsilon + \lambda\tilde{\omega}) \right|_{\lambda=0} = 2 \int_Q e^{-2s\alpha} \phi^{-3} u_\varepsilon \tilde{\omega} dxdt + \frac{2}{\varepsilon} \int_\Omega p_\varepsilon(T)L\tilde{\omega} dx,$$

for all $\tilde{\omega} \in L^2(Q)$.

By the condition of u_ε to be stationary point of $J_\varepsilon(u)$, we must have

$$2 \int_Q e^{-2s\alpha} \phi^{-3} u_\varepsilon \tilde{\omega} dxdt + \frac{2}{\varepsilon} \int_\Omega p_\varepsilon(T)z(T) dx = 0, \tag{4.10}$$

for all $\tilde{w} \in L^2(Q)$ and z weak solution of

$$\begin{cases} z_t - \Delta z + a(t)z = \chi_\omega \tilde{w} & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = 0 & \text{in } \Omega. \end{cases} \quad (4.11)$$

Observe that $z = L\tilde{w}$ and L is a bounded linear function of $\tilde{w} \in L^2(Q)$.

Remark 4.1. As $\tilde{w} \in L^2(Q)$ and $z(x, t) = L\tilde{w}(x, t)$, then $z \in C^0([0, T]; L^2(\Omega))$.

Thus, makes sense $z(x, T) = L\tilde{w}(x, T)$ ■

Multiply (4.11) by w_ε and integrate in Q . We obtain:

$$\begin{aligned} & \int_Q (-w_{\varepsilon t} - \Delta w_\varepsilon + a(t)w_\varepsilon) z \, dxdt \\ & + \int_\Omega z(T)w_\varepsilon(T) \, dx - \int_\Omega z(0)w_\varepsilon(0) \, dx = \int_Q \chi_\omega \tilde{w} w_\varepsilon \, dxdt. \end{aligned} \quad (4.12)$$

Therefore, if w_ε is the weak solution of the problem:

$$\begin{cases} w_{\varepsilon t} + \Delta w_\varepsilon - a(t)w_\varepsilon = 0 & \text{in } Q, \\ w_\varepsilon = 0 & \text{on } \Sigma, \\ w_\varepsilon(x, T) = -\frac{1}{\varepsilon} p_\varepsilon(x, T) & \text{in } \Omega. \end{cases} \quad (4.13)$$

We obtain, from (4.11), (4.12) and (4.13):

$$-\frac{1}{\varepsilon} \int_\Omega p(x, T)z(x, T) \, dx = \int_Q \chi_\omega \tilde{w} w_\varepsilon \, dxdt, \quad (4.14)$$

because $z(x, 0) = 0$ in Ω by (4.11). From (4.10) we modify (4.14) obtaining

$$\int_Q e^{-2s\alpha} \phi^{-3} u_\varepsilon \tilde{w} \, dxdt = \int_Q \chi_\omega \tilde{w} w_\varepsilon \, dxdt,$$

or

$$\int_Q (e^{-2s\alpha} \phi^{-3} u_\varepsilon - \chi_\omega w_\varepsilon) \tilde{w} \, dxdt = 0,$$

for all $\tilde{w} \in L^2(Q)$. It implies that

$$u_\varepsilon = e^{2s\alpha} \phi^3 \chi_\omega w_\varepsilon, \quad \text{a.e. in } Q,$$

with w_ε weak solution of (4.13), i.e., (4.4). This proves Lemma 4.1 ■

Now, we will return to the proof of Theorem 4.1. In fact, the first step, still technic, is to obtain as application of Lemma 4.1 estimates for u_ε and p_ε

to get convergence in order to obtain our objective, which is $p(x, T) = 0$. To this, we multiply both sides of (4.13) by $p_\varepsilon(x, t)$ and integrate in Q .

The second one, we multiply both sides of the system (4.15) below by w_ε and integrate in Q .

$$\begin{cases} p_{\varepsilon t} - \Delta p_\varepsilon + a(t)p_\varepsilon = \chi_\omega w_\varepsilon e^{2s\alpha} \phi^3 & \text{in } Q, \\ p_\varepsilon = 0 & \text{on } \Sigma, \\ p_\varepsilon(x, 0) = p_0(x) & \text{in } \Omega. \end{cases} \tag{4.15}$$

Adding both results, we obtain:

$$\begin{aligned} & \int_{Q_\omega} e^{2s\alpha} \phi^3 w_\varepsilon^2 dxdt = \int_\Omega p_\varepsilon(t)w_\varepsilon(t) dx \Big|_0^T = \\ & - \frac{1}{\varepsilon} \int_\Omega (p_\varepsilon(x, T))^2 dx - \int_\Omega p_\varepsilon(x, 0)w_\varepsilon(x, 0) dx. \end{aligned}$$

After computations we obtain:

$$\int_{Q_\omega} e^{2s\alpha} \phi^3 w_\varepsilon dxdt + \frac{1}{\varepsilon} \int_\Omega p_\varepsilon(x, T)^2 dx = - \int_\Omega p_0(x)w_\varepsilon(x, 0) dx \tag{i}$$

$$\left| \int_\Omega p_0(x)w_\varepsilon(x, 0) dx \right|_{\mathbb{R}} \leq |p_0|_{L^2(\Omega)} |w_\varepsilon(x, 0)|_{L^2(\Omega)}. \tag{ii}$$

By inequality of observability for $w_\varepsilon(x, 0)$, cf. Theorem 3.1, Section 3, we obtain from (ii)

$$\begin{aligned} & \left| \int_\Omega p_0(x)w_\varepsilon(x, 0) dx \right|_{\mathbb{R}} \leq \tag{4.16} \\ & C|p_0|_{L^2(\Omega)} \left(\int_{Q_\omega} e^{2s\alpha} \phi^3 w_\varepsilon^2(x, t) dxdt + \frac{1}{\varepsilon} \int_\Omega p_\varepsilon(x, T)^2 dx \right)^{1/2} \leq \\ & \frac{C^2}{2} |p_0|_{L^2(\Omega)}^2 + \frac{1}{2} \left(\int_{Q_\omega} e^{2s\alpha} \phi^3 w_\varepsilon^2 dxdt + \frac{1}{\varepsilon} \int_\Omega p_\varepsilon(x, T)^2 dx \right). \end{aligned}$$

By (i), (ii) and (4.16), we have

$$\begin{aligned} & \int_{Q_\omega} e^{2s\alpha} \phi^3 w_\varepsilon^2 dxdt + \frac{1}{\varepsilon} \int_\Omega p_\varepsilon(x, T)^2 dx \leq \\ & \frac{C^2}{2} |p_0|_{L^2(\Omega)}^2 + \frac{1}{2} \left(\int_{Q_\omega} e^{2s\alpha} \phi^3 w_\varepsilon^2 dxdt + \frac{1}{\varepsilon} \int_\Omega p_\varepsilon(x, T)^2 dx \right). \end{aligned}$$

From this last inequality $\left(A \leq \frac{C^2}{2} |p_0|^2 + \frac{1}{2} A \right)$, we obtain

$$\int_{Q_\omega} e^{2s\alpha} \phi^3 w_\varepsilon^2 dxdt + \frac{1}{\varepsilon} \int_\Omega p(x, T)^2 dx \leq C|p_0|_{L^2(\Omega)}^2 = \text{constant}. \tag{4.17}$$

Thus from (4.17), we get

$$\int_{\Omega} p_{\varepsilon}(x, T)^2 dx \leq C\varepsilon. \quad (4.18)$$

From (4.18), we have

$$p_{\varepsilon}(x, T) \rightarrow 0 \quad \text{strongly} \quad L^2(\Omega) \quad \text{as} \quad \varepsilon \rightarrow 0.$$

From (4.17), we obtain

$$\int_{Q_{\omega}} e^{2s\alpha} \phi^3 w_{\varepsilon}^2 dxdt < \text{constant}.$$

By Lemma 4.1,

$$u_{\varepsilon} = e^{2s\alpha} \phi^3 \chi_{\omega} w_{\varepsilon} \quad \text{a.e. in } Q,$$

or

$$\int_Q u_{\varepsilon}^2 dxdt \leq C_0 \int_Q e^{2s\alpha} \phi^3 \chi_{\omega} w_{\varepsilon}^2 dxdt < C.$$

Thus, we have

$$u_{\varepsilon} \rightharpoonup u \quad \text{weakly in } L^2(Q).$$

From (4.15), we obtain

$$p_{\varepsilon} \rightharpoonup p \quad \text{weakly} \quad H^1(0, T; H^{-1}(\Omega)) \cap L^2(0, T; H_0^1(\Omega)),$$

then

$$p_{\varepsilon} \rightharpoonup p \quad \text{strongly} \quad C^0([0, T]; L^2(\Omega)). \quad (4.19)$$

From (4.18), we obtain

$$p_{\varepsilon}(x, T) \rightarrow 0 \quad \text{strongly} \quad L^2(\Omega),$$

whence there exists a subsequence of $p_{\varepsilon}(x, T)$ such that

$$p_{\varepsilon}(x, T) \rightarrow 0 \quad \text{a.e. in } \Omega. \quad (4.20)$$

From (4.19), we have

$$p_{\varepsilon}(x, t) \rightarrow p(x, t) \quad \text{a.e. in } \Omega,$$

for $0 \leq t \leq T$. Then

$$p_{\varepsilon}(x, T) \rightarrow p(x, T) \quad \text{a.e. in } \Omega.$$

By (4.20) we have $p(x, T) = 0$ a.e. in Ω . Observe that $u_\varepsilon = w_\varepsilon e^{2s\alpha} \phi^3$ in the system (4.15). Thus, when $\varepsilon \rightarrow 0$ in (4.15) we obtain a control $u \in L^2(Q)$ and a function

$$p \in H^1(0, T; H^{-1}(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$$

solution, in weak sense, of

$$\begin{cases} p_t - \Delta p + a(t)p = \chi_\omega u & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(x, 0) = p_0 & \text{in } \Omega, \end{cases}$$

such that

$$p(x, T) = 0 \quad \text{a.e. in } \Omega.$$

This conclusion proves Theorem 4.1 ■

5 Null controllability: Nonlinear State Equation

We now investigate null controllability for nonlinear state equation:

$$\begin{cases} p_t - \Delta p + g(p) = \chi_\omega u & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(x, 0) = p_0(x) & \text{in } \Omega. \end{cases} \quad (5.1)$$

We suppose $g: \mathbb{R} \rightarrow \mathbb{R}$, globally Lipschitz and $g(0) = 0$. It means,

$$|g(p_1) - g(p_2)| \leq M|p_1 - p_2| \quad \text{for all } p_1, p_2 \in \mathbb{R} \quad \text{and } M \text{ constant.}$$

We define the function $f: \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$f(p) = \begin{cases} \frac{g(p)}{p} & \text{if } |p| > 0, \\ \lim_{\varepsilon \rightarrow 0} \frac{g(\varepsilon)}{\varepsilon} & \text{if } p = 0. \end{cases}$$

We introduce the Hilbert space

$$W^1(0, T; H_0^1(\Omega), H^{-1}(\Omega)) = \{p \in L^2(0, T; H_0^1(\Omega)), p_t \in L^2(0, T; H^{-1}(\Omega))\},$$

with the norm

$$\|p\|_{W^1}^2 = \|p\|_{L^2(0, T; H_0^1(\Omega))}^2 + \|p_t\|_{L^2(0, T; H^{-1}(\Omega))}^2.$$

We have

$$W^1(0, T; H_0^1(\Omega), H^{-1}(\Omega)) \subset C^0([0, T]; L^2(\Omega)) \subset L^2(Q).$$

We consider the subset B of $L^2(Q)$ defined by

$$B = \{b \in W^1(0, T; H_0^1(\Omega), H^{-1}(\Omega)); \|b\|_{W^1} < M_1\}.$$

For $\bar{p} \in B$, $p_0 \in L^2(\Omega)$, $u \in L^2(Q)$, we consider the linear state equation

$$\begin{cases} p_t - \Delta p + f(\bar{p})p = \chi_\omega u & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(x, 0) = p_0(x) & \text{in } \Omega. \end{cases} \quad (5.2)$$

Observe that (5.2) is a linearization of (5.1). Also note that $a(x, t) = f(\bar{p}(x, t))$ with $\bar{p} \in B$ a ball of W^1 . We have $|a(x, t)| < M$. Thus, for $\bar{p} \in W^1$, for $T > 0$, there exists $u \in L^2(Q, e^{-2s\alpha} \phi^{-3})$ such that the weak solution p of (5.2) is satisfies $p(x, T) = 0$ in Ω , that is, we have null controllability for (5.2).

Theorem 5.1. *Suppose $g: \mathbb{R} \rightarrow \mathbb{R}$, globally Lipschitz and $g(0) = 0$, $p_0 \in L^2(\Omega)$ and $T > 0$. There exists $u \in L^2(Q, e^{-2s\alpha} \phi^{-3})$ and $p \in W^1(0, T; H_0^1(\Omega), H^{-1}(\Omega))$, weak solution of (5.1) such that $p(x, T) = 0$ in Ω .*

Proof. We apply fixed point method as is usually done. As we will work with multi-valued mapping we need a infinity dimensional version of Shizuo Kakutani fixe point theorem. Among many generalization we employ Glicksberg [9] version, see also Browder [4], which is the following.

“Let B be a non-empty convex, compact subset of a locally convex topological vector space X and Φ a mapping which takes $p \in B$ into a non-empty subset $\Phi(p)$ of X , such that is convex, compact and has closed graphic. Then the set of fixe point of Φ is non-empty and compact.”

In our case we have $X = L^2(Q)$ and

$$B = \{b \in W^1(0, T; H_0^1(\Omega), H^{-1}(\Omega)); \|b\|_{W^1} < M_1\} \subset C^0([0, T]; L^2(\Omega)) \subset L^2(Q),$$

the constant $M_1 > 0$ is obtained in (5.6). Observe that B is a convex set of $L^2(Q)$. Let us prove that B is a compact set of $L^2(Q)$. In fact, let $(b_n)_{n \in \mathbb{N}}$ be a sequence of $b_n \in B$. Then $\|b\|_{W^1} < M$, that is, $(b_n)_{n \in \mathbb{N}}$ is bounded in

$L^2(0, T; H_0^1(\Omega))$ and $\frac{db_n}{dt}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$. By the theorem of compacticity of Aubin [1], it follows

$$b_n \rightarrow b \quad \text{strongly in } L^2(Q).$$

We define the mapping Φ in B as follows: for $\bar{p} \in B$, we set

$$\Phi(\bar{p}) = \left\{ p \in W^1(0, T; H_0^1(\Omega), H^{-1}(\Omega)), \text{ weak solution of (5.2) for} \right. \\ \left. u \in L^2(Q, e^{-2s\alpha} \phi^{-3}) \text{ with } \int_Q e^{-2s\alpha} \phi^{-3} u^2 dxdt \leq C \int_\Omega p_0^2 dx \text{ such that } p(x, T) = 0 \right\}$$

Remark 5.1. Observe that for $\bar{p} \in B$, $a(x, t) = f(\bar{p}(x, t))$ is bounded by definition of f , because g is Lipschitz and $g(0) = 0$. Thus for $u \in L^2(Q, e^{-2s\alpha} \phi^{-3})$, $p_0 \in L^2(\Omega)$ there exists $p \in W^1(0, T; H_0^1(\Omega), H^{-1}(\Omega))$ solution of (5.2) such that $p(x, T) = 0$, cf. Section 4. Thus, when $\bar{p} \in B$, $\Phi(\bar{p})$ is not empty in $L^2(Q)$ ■

Thus, when $u \in L^2(Q, e^{-2s\alpha} \phi^{-3})$, $p_0 \in L^2(\Omega)$, $\bar{p} \in B$, $\Phi(\bar{p})$ is a subset of $L^2(Q)$, that is, Φ is a multi-valued mapping. A fixe point of $\Phi(\bar{p})$ is a vector $\bar{p} \in B$ such that $\bar{p} \in \Phi(\bar{p})$. Thus, this fixed point \bar{p} is solution of (5.2) with $\bar{p}(x, T) = 0$, that is, \bar{p} is solution of (5.1) with $\bar{p}(x, T) = 0$, which implies null controllability for (5.1).

Thus, $\Phi: B \rightarrow 2^B$ and we prove that it has a fixed point. We must prove, see Glicksberg [9], that $\Phi(\bar{p})$ is non-empty, $\Phi(B) \subset B$ and Φ is closed.

(i) $\Phi(\bar{p})$ is non-empty for $\bar{p} \in B$, already proved.

(ii) $\Phi(B) \subset B$. In fact, for all $\bar{p} \in B$ if $p \in \Phi(\bar{p})$, by definition $\Phi(\bar{p})$, p is weak solution of (5.2). Multiplying both sides of (5.2)₁ by p and integrate in Ω , we get

$$\frac{d}{dt} |p(t)|_{L^2(\Omega)}^2 + \|p(t)\|_{H_0^1(\Omega)}^2 \leq M |p(t)|_{L^2(\Omega)}^2 + |\chi_\omega|_{L^2(Q)} |u|_{L^2(Q)}.$$

Integrating on $[0, t)$, we have

$$|p(t)|_{L^2(\Omega)}^2 + \int_0^t \|p(s)\|_{H_0^1(\Omega)}^2 ds \leq |p_0|_{L^2(\Omega)}^2 + |u|_{L^2(Q)}^2 \\ + \left(M + \frac{1}{4} \right) \int_0^t |p(s)|_{L^2(\Omega)}^2 ds.$$

By Gronwall inequality, it yields

$$|p(t)|_{L^2(\Omega)}^2 + \int_0^T \|p(t)\|_{H_0^1(\Omega)}^2 dt \leq \left(|p_0|_{L^2(\Omega)}^2 + C_1 |p_0|_{L^2(\Omega)}^2 \right) e^{(M+\frac{1}{4})T} = C_2. \tag{5.3}$$

We also have, for all $v \in H_0^1(\Omega)$ with $\|v\|_{H_0^1(\Omega)} \leq 1$, that

$$\begin{aligned} |\langle p_t, v \rangle| &= |\langle +\Delta p - f(\bar{p})p + \chi_\omega u, v \rangle| \leq \\ &\|p(t)\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} + M|p(t)|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + |u|_{L^2(Q)} |v(t)|_{L^2(\Omega)} \leq \\ &\left(\|p(t)\|_{H_0^1(\Omega)} + M C_2^{1/2} C_3 + C_1^{1/2} |p_0|_{L^2(\Omega)} C_3 \right) \|v\|_{H_0^1(\Omega)}, \end{aligned} \tag{5.4}$$

where C_3 is the constant of immersion of $H_0^1(\Omega)$ into $L^2(\Omega)$. Thus, we obtain

$$\|p_t(t)\|_{H^{-1}(\Omega)} \leq \|p(t)\|_{H_0^1(\Omega)} + C_4,$$

with $C_4 = M C_2^{1/2} C_3 + C_1^{1/2} |p_0|_{L^2(\Omega)} C_3$. It follows that

$$\|p_t\|_{L^2(0,T;H^{-1}(\Omega))}^2 \leq 2\|p\|_{L^2(0,T;H_0^1(\Omega))}^2 + 2C_4^2 T = 2C_2 + 2C_4^2 T. \tag{5.5}$$

From (5.3) and (5.5), we obtain

$$\int_0^T \|p(t)\|_{H_0^1(\Omega)}^2 dt + \int_0^T \|p_t(t)\|_{H^{-1}(\Omega)} \leq M_1^2. \tag{5.6}$$

Thus,

$$\|p\|_{W^1} \leq M_1, \text{ with } M_1 = 2C_2 + 2C_4^2 T^{1/2}.$$

Therefore, if $\bar{p} \in B$ then $\Phi(\bar{p}) \subset B$. Thus, $\Phi: B \rightarrow 2^B$.

(iii) $\Phi(\bar{p})$ is closed in $L^2(Q)$. In fact, let \bar{p} be in B fixed and $p_n \in \Phi(\bar{p})$ such that

$$p_n \rightarrow p \text{ strongly in } L^2(Q).$$

By definition of $\Phi(\bar{p})$, we have

$$\begin{cases} p_{nt} - \Delta p_n + f(\bar{p})p_n = \chi_\omega u_n & \text{in } Q, \\ p_n = 0 & \text{on } \Sigma, \\ p_n(0) = p_0 & \text{in } \Omega, \end{cases} \tag{5.7}$$

with

$$\int_Q e^{-2s\alpha} \phi^{-3} u_n^2(x, t) dxdt \leq C \int_\Omega p_0^2(x) dx,$$

which implies $\|u_n\|_{L^2(Q)}^2 \leq C\|p_0\|_{L^2(\Omega)}^2$. Thus, we extract a subsequence of $(u_n)_{n \in \mathbb{N}}$, which will also be denote by $(u_n)_{n \in \mathbb{N}}$, such that

$$u_n \rightharpoonup u \quad \text{weakly in } L^2(Q). \tag{5.8}$$

By the same argument to obtain (5.6) from (5.4) and (5.5), we get

$$\|p_n\|_{L^2(0,T;H_0^1(\Omega))}^2 + \|p_{nt}\|_{L^2(0,T;H^{-1}(\Omega))}^2 \leq M_1^2. \tag{5.9}$$

From (5.9) we extract a subsequence $(p_n)_{n \in \mathbb{N}}$, such that

$$\left\{ \begin{array}{l} p_n \rightharpoonup p \quad \text{weakly } L^2(0, T; H_0^1(\Omega)), \\ p_{nt} \rightharpoonup p_t \quad \text{weakly } L^2(0, T; H^{-1}(\Omega)), \\ p_n \rightharpoonup p \quad \text{strongly } L^2(Q). \end{array} \right. \tag{5.10}$$

The last convergence has been obtained compactness theorem, cf Aubin [1]. From (5.10) we pass to the limits in (5.7), as $n \rightarrow \infty$, to obtain

$$\left\{ \begin{array}{l} p_t - \Delta p + f(\bar{p})p = \chi_\omega u \quad \text{in } Q, \\ p = 0 \quad \text{on } \Sigma. \\ p(x, 0) = p_0(x) \quad \text{in } \Omega, \end{array} \right. \tag{5.11}$$

and

$$\int_Q e^{-2s\alpha} \phi^{-3} u^2(x, t) dxdt \leq C \int_\Omega p_0(x)^2 dx.$$

Thus, $p \in \Phi(\bar{p})$ and $\Phi(\bar{p})$ is closed. Therefore, since B is compact of $L^2(Q)$ and $\Phi(\bar{p}) \subset B$ is closed, it implies that $\Phi(\bar{p})$ is a compact of $L^2(Q)$.

(iv) Φ has the closed graph in $L^2(Q) \times L^2(Q)$.

Remark 5.2. *Let X be a locally convex topological vector space and $B \subset X$ and the mapping*

$$\Phi: B \rightarrow X$$

which for each $\bar{p} \in B$ corresponds a non void convex set $\Phi(\bar{p})$ of X .

We say that Φ is closed if its graph

$$\bigcup_{\bar{p} \in B} (\bar{p}, \Phi(\bar{p}))$$

is a closed subset of the Cartesian product $X \times X$.

In terms of direct sets it may be stated as follows:

$$x_\varepsilon \rightarrow x \text{ in } X, \quad y_\varepsilon \in \Phi(x_\varepsilon) \quad \text{and} \quad y_\varepsilon \rightarrow y. \quad \text{Then} \quad y \in \Phi(x).$$

This argument generalizes the terminology for closed operator A with A a function with domain $D(A)$ dense in X . In fact, we say that $A: D(A) \rightarrow X$ is a closed operator, when

$$x_n \rightarrow x \quad \text{and} \quad Ax_n \rightarrow y,$$

then $x \in D(A)$ and $y = Ax$. This means that the graph of A is closed in $X \times X$ ■

Let us prove (iv). In fact, let \bar{p}_n, p_n be such that

$$\bar{p}_n \rightarrow \bar{p}, p_n \rightarrow p \quad \text{strongly} \quad L^2(Q), \tag{5.12}$$

and $p_n \in \Phi(\bar{p}_n)$. We must prove that $p \in \Phi(\bar{p})$. In fact, from $p_n \in \Phi(\bar{p}_n)$, it follows that p_n is weak solution of:

$$\left\{ \begin{array}{l} p_{nt} - \Delta p_n + f(\bar{p}_n)p_n = \chi_\omega u_n \quad \text{in } Q, \\ p_n = 0 \quad \text{on } \Sigma, \\ p_n(x, 0) = p_0(x) \quad \text{in } \Omega, \end{array} \right. \tag{5.13}$$

and

$$\int_Q e^{-2s\alpha} \phi^{-3} u_n^2(x, t) dxdt \leq C \int_\Omega p_0(x)^2 dx.$$

See definition of $\Phi(\bar{p})$. By the same argument to obtain (5.9), applied to (5.13), we obtain

$$\|p_n\|_{L^2(0,T;H_0^1(\Omega))}^2 + \|p_{nt}\|_{L^2(0,T;H^{-1}(\Omega))}^2 \leq M_1^2. \tag{5.14}$$

From (5.14) and the estimate for u_n in $L^2(Q)$, since $e^{-2s\alpha} \phi^{-3} \geq C$, we obtain subsequences $(u_n)_{n \in \mathbb{N}}$ and $(p_{nt})_{n \in \mathbb{N}}$ such that

$$\left\{ \begin{array}{l} u_n \rightharpoonup u \quad \text{weakly} \quad L^2(\Omega), \\ p_n \rightharpoonup p \quad \text{weakly} \quad L^2(0, T; H_0^1(\Omega)), \\ p_{nt} \rightharpoonup p_t \quad \text{weakly} \quad L^2(0, T; H^{-1}(\Omega)). \end{array} \right. \tag{5.15}$$

From (5.12) we obtain subsequences $(\bar{p}_n)_{n \in \mathbb{N}}, (p_n)_{n \in \mathbb{N}}$ such that

$$\left\{ \begin{array}{l} \bar{p}_n \rightarrow \bar{p} \quad \text{a.e. in } Q, \\ p_n \rightarrow p \quad \text{a.e. in } Q. \end{array} \right. \tag{5.16}$$

By continuity of $f: \mathbb{R} \rightarrow \mathbb{R}$ we obtain $f(\bar{p}_n) \rightarrow f(\bar{p})$ a.e. in Q , then by (5.16), we have

$$f(\bar{p}_n)p_n \rightarrow f(\bar{p})p \quad \text{a.e. in } Q. \quad (5.17)$$

We also have

$$\int_Q |f(\bar{p}_n)p_n|^2 dxdt \leq M \int_Q |p_n|^2 dxdt \leq CM. \quad (5.18)$$

Thus, by Lions [12]- Lemma 3 we obtain from (5.17) and (5.18), that

$$f(\bar{p}_n)p_n \rightharpoonup f(\bar{p})p \quad \text{weakly } L^2(Q). \quad (5.19)$$

Thus, passing to the limits in (5.13) as $n \rightarrow \infty$, we obtain:

$$\begin{cases} p_t - \Delta p + f(\bar{p})p = \chi_\omega u & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(x, 0) = p_0(x) & \text{in } \Omega, \end{cases} \quad (5.20)$$

and

$$\int_Q e^{-2s\alpha} \phi^{-3} u^2(x, t) dxdt \leq C \int_\Omega p_0(x)^2 dx,$$

what proves that $p \in \Phi(\bar{p})$.

Conclusion. The multi-valued mapping $\Phi: B \rightarrow 2^B$ satisfies the conditions of the infinity dimensional version of Shizuo Kakutani [10], cf. Glicksberg [9], thus it has a fixe point, that is, there exists $\bar{p} \in \Phi(\bar{p})$. It proves null controllability for the nonlinear state equation (5.1). The proof of Theorem 5.1 is complete ■

6 Approximate controllability

We consider the linear parabolic state system

$$\begin{cases} p_t - \Delta p + a(t)p = \chi_\omega u & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(x, 0) = p_0 & \text{in } \Omega. \end{cases} \quad (6.1)$$

As we have defined, in the Introduction, Section 1, (ii), we say that (6.1) is approximate controllable in $L^2(\Omega)$, at time $T > 0$, if for each $\varepsilon > 0$, given

$p_0 \in L^2(\Omega)$ and $p_T \in L^2(\Omega)$, there exists a control $u \in L^2(Q_\omega)$, $Q_\omega = \omega \times (0, T)$ such that the corresponding solution of $p(x, t)$ of (6.1) satisfies:

$$|p(x, T) - p_T(x)|_{L^2(\Omega)} < \varepsilon. \quad (6.2)$$

This concept of approximate controllability was introduced by J. L. Lions [11] employing a theorem of continuation by Mizohata, see also Cara-Guerreiro [5], Fabre-Puel-Zuazua [8], Zuazua [13].

In this section we prove the same result as an application of Carleman Inequalities.

Theorem 6.1. *Fixe $T > 0$ and given $\varepsilon > 0$ and $p_0, p_T \in L^2(\Omega)$. Then, there exists a control $u \in L^2(Q_\omega)$ such that the solution p of the state equation (6.1) satisfies (6.2).*

Proof. As the system is linear we can suppose $p_0 = 0$. In fact, with $p_0 \in L^2(\Omega)$ we solve the problem:

$$\begin{cases} \widehat{p}_t - \Delta \widehat{p} + a\widehat{p} = 0 & \text{in } Q, \\ \widehat{p} = 0 & \text{on } \Sigma, \\ \widehat{p}(0) = p_0 & \text{in } \Omega. \end{cases} \quad (6.3)$$

Thus, if $w = p - \widehat{p}$, w is solution of (6.1) with $w(0) = 0$. Therefore, we consider (6.1) but with $p_0 = 0$. To prove approximate controllability we define the set:

$$R_L(T) = \{p(x, T); p \text{ solution of (6.1), with } u \in L^2(Q_\omega)\}.$$

This set is called reachable set and the index L means for linear problem. To prove approximate controllability it is sufficient to prove that $R_L(T)$ is dense in $L^2(\Omega)$. We will prove reasoning by contradiction.

Suppose $R_L(T)$ is not dense in $L^2(\Omega)$. Thus, there exists a non null vector w_T in the orthogonal complement $R_L(T)^\perp$ in $L^2(\Omega)$. With w_T we consider the adjoint state:

$$\begin{cases} w_t + \Delta w - a(t)w = 0 & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(x, T) = w_T & \text{in } \Omega. \end{cases} \quad (6.4)$$

Multiply (6.4) by p , solution of (6.1) with $u \in L^2(Q_\omega)$ and integrate in Q . We obtain:

$$\int_Q (w_t - \Delta w + a(t)w)p \, dxdt = 0.$$

Then,

$$\int_Q (p_t - \Delta p + a(t)p)w \, dxdt + \int_\Omega w(T)p(T)dx - \int_\Omega w(0)p(0)dx = 0.$$

Observe that p satisfies (6.1)₁, $w(T) = w_T$ and $p(0) = p_0 = 0$. Thus, we obtain:

$$- \int_{Q_\omega} u(x, t)w(x, t) \, dxdt + \int_\Omega w_T(x)p(x, T) \, dx = 0.$$

To analyse the second integral above, observe that $w_T(x)$ belongs to the orthogonal $R_L(T)^\perp$ and $p(x, T)$ belongs to $R_L(T)$. Thus, the second integral is zero. We obtain

$$\int_{Q_\omega} u(x, t)w(x, t) \, dxdt = 0 \quad \text{for all } u \in L^2(Q_\omega),$$

what implies $w(x, t) = 0$ a.e. in Q_ω . By Carleman Inequality, cf. Section 2, with $f = 0$, we obtain

$$\int_Q (s^3 \phi^3 w(x, t)^2) e^{2s\alpha} \, dxdt \leq 0. \quad (6.5)$$

We have $s^3 \phi^3 \geq C > 0$, $e^{2s\alpha} > 0$ in $\Omega \times (0, T)$. Then it implies $w = 0$ a.e. in Q , by (6.5). We then have $w(x, T) = w_T(x) = 0$, which is a contradiction. Thus $R_L(T)$ is dense in $L^2(\Omega)$ ■

7 APPENDIX

Observe that $\varphi = e^{\lambda|\psi|}$ is constant for $x \in \bar{\Omega}$ and

$$e^{2s\alpha(x, t)} \leq e^{-\frac{2s\varphi}{\beta(t)}}, \quad 0 < t < T.$$

Then, we have

$$\begin{aligned} & \int_{t_1}^{t_2} \left[\int_0^t \left(\int_\Omega e^{2s\alpha(x, t)} |f_1(x, y)|^2 \, dx \right) dy \right] dt \leq \\ & \int_{t_1}^{t_2} \left[\int_0^t \left(\int_\Omega e^{-2s\varphi/\beta(t)} |f_1(x, y)|^2 \, dx \right) dy \right] dt \leq \\ & \int_0^T \left[\int_0^t \left(\int_\Omega e^{-2s\varphi/\beta(t)} |f_1(x, y)|^2 \, dx \right) dy \right] dt = \\ & \int_\Omega \left[\int_0^T \left(\int_0^t e^{-2s\varphi/\beta(t)} |f_1(x, y)|^2 \, dy \right) dt \right] dx. \end{aligned}$$

Now we will prove the following inequality.

$$\int_{\Omega} \left[\int_0^T \left(\int_0^t e^{-2s\varphi/\beta(t)} |f_1(x, y)|^2 dy \right) dt \right] dx \leq C \int_0^T \int_{\Omega} |f_1(x, y)|^2 dx dy.$$

Analysis of the integral

$$I = \int_0^T \left(\int_0^t e^{-2s\varphi/\beta(t)} |f_1(x, y)|^2 dy \right) dt.$$

In the integral I , we have

$$0 < y < t \quad \text{and} \quad 0 < t < T.$$

Let us consider the change of variables in I , defined by the linear mapping $\sigma(t, y) = (y, t)$, an involution, from \mathbb{R}^2 into \mathbb{R}^2 . It is given by

$$(at + by, ct + dy) = (y, t),$$

with $a = 0$, $b = 1$, $c = 1$, $d = 0$. The matrix of σ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $|\det \sigma| = 1$.

Let us consider the domain K of \mathbb{R}^2 defined by:

$$K = \{(t, y); 0 < t < T, y < t\}$$

and $\widehat{K} = \sigma(K)$ is defined by:

$$\widehat{K} = \{(y, t); 0 < y < T, y > t\}.$$

We have

$$I = \int_0^T \left(\int_0^t e^{-2s\varphi/\beta(t)} |f_1(x, y)|^2 dy \right) dt = \int_0^T \left(\int_y^T e^{-2s\varphi/\beta(y)} |f_1(x, t)|^2 dt \right) dy.$$

We have regularity for $e^{-2s\varphi/\beta(t)}$. Then:

$$\begin{aligned} & \int_0^T \left(\int_0^t e^{-2s\varphi/\beta(t)} |f_1(x, y)|^2 dy \right) dt \leq \int_0^T \left(\int_0^T e^{-2s\varphi/\beta(y)} |f_1(x, t)| dt \right) dy \\ & = \left(\int_0^T |f_1(x, t)|^2 dt \right) \left(\int_0^T e^{-2s\varphi/\beta(y)} dy \right) = C \int_0^T |f_1(x, t)| dt, \end{aligned}$$

with $C > 0$ depending of Ω and T . Integrating on Ω , we have

$$\int_{\Omega} \left(\int_0^T \left(\int_0^t e^{-2s\varphi/\beta(t)} |f_1(x, t)|^2 dy \right) \right) dt dx \leq C \int_0^T \int_{\Omega} |f_1(x, t)|^2 dx dt \quad \blacksquare$$

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“Porque de feitos tais, por mais que diga,
Mais me h de ficar ainda por dizer”

Cames (Lusadas-Canto III)

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