

A superlinear type problem for a p-laplacian perturbation

F. O. de Paiva and H. R. Quoirin *

Dedicated to Antonio Gervásio Colares on the occasion of his 80th birthday.

Abstract

In this work we investigate existence and multiplicity of positive solutions for the superlinear type problem

$$\begin{cases} -\Delta_p u + V(x)u^{p-1} = f(x)u^{q-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $q > p > 1$ and f changes sign.

1 Introduction

Let Ω be a bounded domain of \mathbb{R}^N , $2 < q < 2^*$ and $f : \Omega \rightarrow \mathbb{R}$ be a bounded sign-changing function. Unless otherwise stated we assume that $\partial\Omega$ is C^2 . It is a standard fact that the semilinear equation

$$(I) \quad \begin{cases} -\Delta u = f(x)u^{q-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a positive solution. Such existence result persists if (I) is linearly perturbed, giving rise to:

$$(II) \quad \begin{cases} -\Delta u - \lambda u = f(x)u^{q-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The perturbation has to be accorded with λ_1 , the first eigenvalue of $-\Delta$, in the following sense:

2000 *AMS Subject Classification.* 35J25 (58E05)

Key Words and Phrases: quasilinear problems, indefinite weight, multiplicity of solution.

*This work was supported by CNPq/Brazil - FNRS/Belgium

- if $\lambda < \lambda_1$ then the positive definiteness of the left-hand side is preserved and (II) has a positive solution.
- if $\lambda > \lambda_1$ then the functional $\int_{\Omega} (|\nabla u|^2 - \lambda|u|^2)$ is no longer coercive, but (II) keeps the positive solution originated from the coercive case provided $\lambda < \lambda^*$, for some $\lambda^* > \lambda_1$. Furthermore, a second positive solution arises if $\int_{\Omega} f\varphi_1^q < 0$.
- if $\lambda > \lambda^*$ then (II) has no positive solution.

The result described above was established in several works devoted to semilinear equations involving indefinite superlinearities, starting from Ouyang [13] and passing by Alama-Tarantello [1], Del Pino [9], Berestycki-Capuzzo-Dolcetta-Nirenberg [4], Terhani [14] and Chabrowski-Marcos do Ó [6].

One may then consider the quasilinear version of (II), namely,

$$(P_{\lambda}) \quad \begin{cases} -\Delta_p u - \lambda u^{p-1} = f(x)u^{q-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < q \leq p^*$. This problem was investigated by Ilyasov [12] and Birindelli-Demengel [5] (see also Drabek-Huang [10] for a similar problem with $\Omega = \mathbb{R}^N$). Both works extended partially the above result to (P_{λ}) . Here we complete the result, see Theorem 1.7. A formulation for the optimal value of λ^* was also given by Ilyasov.

Our purpose here is to treat the ‘no parameter’ version of (P_{λ}) :

$$(P) \quad \begin{cases} -\Delta_p u + V(x)u^{p-1} = f(x)u^{q-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

We assume that $1 < p < q < p^*$, $f \in L^{\infty}(\Omega)$ and $V \in L^r(\Omega)$ with $r > \frac{N}{p}$ if $p < N$ and $r > 1$ if $p \geq N$. We aim at obtaining existence and multiplicity of solutions for (P) according to the sign of

$$\lambda_1(V) := \min \left\{ \int_{\Omega} (|\nabla u|^p + V|u|^p); \|u\|_p = 1 \right\},$$

the first eigenvalue of $-\Delta_p + V$, and $\int_{\Omega} f\varphi_V^q$, where φ_V is the first positive eigenfunction L^p -normalized. Notice that when $V \equiv -\lambda$ one has $\varphi_V \equiv \varphi_1$, but, in general, φ_V depends on V .

We describe now our approach and results.

Let the functionals E_V and F be defined on $W_0^{1,p}(\Omega)$ by

$$E_V(u) = \int_{\Omega} (|\nabla u|^p + V(x)|u|^p), \quad F(u) = \int_{\Omega} f(x)|u|^q.$$

It is straightforward that F is weakly continuous and that E_V is weakly lower semi-continuous. Moreover, E_V is coercive if and only if $\lambda_1(V) > 0$.

Our approach is prompted by [7], where the indefinite eigenvalue problem

$$(P_{V,m}) \quad -\Delta_p u + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u, \quad u \in W_0^{1,p}(\Omega)$$

is studied. When m is sign-changing, a natural way to obtain positive solutions for this equation is to consider the minimization problems

$$\min\{E_V(u); \int_{\Omega} m|u|^p = \pm 1\}.$$

It is shown that these minima are achieved if and only if $\alpha_1(V, m) \geq 0$, where

$$\alpha_1(V, m) := \min\{E_V(u); \int_{\Omega} m|u|^p = 0 \text{ and } \|u\|_p = 1\}.$$

This condition comes from the relation

$$\alpha_1(V, m) = \max_{t \in \mathbb{R}} \mu_1(t)$$

where $\mu_1(t) = \min\{E_V(u) - t \int_{\Omega} m|u|^p; \|u\|_p = 1\}$. One can easily see that the zeros of μ_1 provide all the principal eigenvalues of $(P_{V,m})$. Properties of μ_1 such as concavity and decaying to $-\infty$ when $t \rightarrow \pm\infty$ yield then the existence of principal eigenvalues, given by

$$\lambda_1(V, m) := \inf \left\{ E_V(u); \int_{\Omega} m|u|^p = 1 \right\}$$

and

$$\lambda_{-1}(V, m) := \inf \left\{ E_V(u); \int_{\Omega} m|u|^p = -1 \right\}.$$

The lack of homogeneity in (P) prevents us from setting $\mu(t) := \min\{E_V(u) - t \int_{\Omega} f|u|^q; \|u\|_p = 1\}$. However, several conclusions similar to those holding for $(P_{V,m})$ can be deduced by dealing with

$$\alpha(V) = \alpha(V, f) := \min \left\{ E_V(u); \int_{\Omega} f|u|^q = 0 \text{ and } \|u\|_p = 1 \right\}.$$

Indeed, one is naturally led to consider

$$c_1 := \inf \left\{ E_V(u); \int_{\Omega} f|u|^q = -1 \right\} \quad \text{and} \quad c_2 := \inf \left\{ E_V(u); \int_{\Omega} f|u|^q = 1 \right\}.$$

We can prove that these infima are achieved and that at least one of them provides a solution when $\int_{\Omega} f\varphi_V^q < 0$ and $\alpha(V, f) > 0$. More generally:

Theorem 1.1. *Problem (P) has a solution if either $\lambda_1(V) > 0$ or $\lambda_1(V) \leq 0$ and $F(\varphi_V) < 0 < \alpha(V, f)$. Moreover, if $\lambda_1(V) < 0$ and $F(\varphi_V) < 0 < \alpha(V, f)$ then (P) has at least two solutions. If $F(\varphi_V) \geq 0 \geq \lambda_1(V)$ then (P) has no solution.*

This theorem summarizes the following results:

Theorem 1.2. *Assume that $\alpha(V) > 0$.*

1. *If $\lambda_1(V) > 0$ then (P) has at least one solution.*
2. *If $\lambda_1(V) < 0$ and $F(\varphi_V) < 0$ then (P) has at least two solutions.*

Theorem 1.2 is the analogue version of the classical statement that holds for the semilinear case. The assumption $\alpha(V) > 0$, similar to $\lambda < \lambda^*$, guarantees that c_1 and c_2 are well-defined.

When dropping the assumption $\alpha(V) > 0$, the second statement in Theorem 1.2 can be proved provided $\lambda_1(V)$ is close to zero. To this end, we need to assume a L^∞ a priori bound on V and that $F(\varphi_V)$ is negative and away from zero.

Corollary 1.3. *Let $\delta, R > 0$ be fixed. Then there exists $\varepsilon = \varepsilon(\delta, R) > 0$ such that (P) has at least two solutions if $\|V\|_r \leq R$, $-\varepsilon < \lambda_1(V) < 0$, and $F(\varphi_V) \leq -\delta$.*

The ‘borderline case’ $\alpha(V) = 0$ stands for the case $\lambda = \lambda^*$ in (II):

Proposition 1.4. *If $f \neq 0$ a.e., $\alpha(V) = 0$, and $\int f\varphi_V^q < 0$ then (P) has at least one solution.*

The non-existence for (P) holds as follows:

Theorem 1.5. *If $\int f\varphi_V^q = 0 = \lambda_1(V)$ then (P) has no solution.*

Theorem 1.6. *Given $\delta, R > 0$ there exists $\varepsilon = \varepsilon(\delta, R) > 0$ such that (P) has no solution if $\|V\|_\infty \leq R$, $\lambda_1(V) \leq -\delta$ and $\int f|\varphi_V|^q \geq \varepsilon$.*

Finally, going back to the parameter version of (P), it is possible to get a better description of the solution set:

Theorem 1.7. *Assume that $p < q < p^*$, $\int f\varphi_1^q < 0$ and either that Ω has a C^2 boundary or that $p \geq 2$. Then there exists $\lambda^* > \lambda_1$ such that (P_λ) has at least two solutions if $\lambda_1 < \lambda < \lambda^*$. If $\lambda > \lambda^*$ then (P_λ) has no solution.*

The paper is organized as follows: in section 2, we prove the existence statements for (P) . Section 3 is devoted to the non-existence results. In Section 4, we focus on the parameter version of (P) .

2 Existence of solutions

We collect now some results in order to prove Theorem 1.2.

Some Lemmata

Lemma 2.1. $\alpha(V)$ is achieved.

Proof. We make use of the following inequality (see [7]):

$$\|u\|^p \leq C_1 E_V(u) + C_2 \|u\|_p^p \tag{2.1}$$

for every $u \in W_0^{1,p}(\Omega)$ and some $C_1, C_2 > 0$ depending only on V . Hence if u_n is a minimizing sequence for $\alpha(V, f)$, it is bounded and, up to a subsequence, it converges weakly to some u_0 . By a standard compactness argument, u_0 realizes $\alpha(V, f)$. \square

From its definition it is clear that $\alpha(V) \geq \lambda_1(V)$ and that $\alpha(V) > \lambda_1(V)$ if $\int f|\varphi_V|^q < 0$.

The next lemma gives a necessary relation between the first eigenpair $(\lambda_1(V), \varphi_V)$ and the solutions of (P) .

Lemma 2.2. If u solves (P) then $\int f\varphi_V^q \leq \lambda_1(V) \int \varphi_V^q u^{p-q}$.

Proof. We proceed as in [5, proof of Theorem 1.2]. Let us assume that u is a solution of $(P_{V,f})$. Hence, by the strong maximum principle in [17], we have $u > 0$ on Ω . Moreover, by the Hopf lemma, $\frac{\partial u}{\partial \nu} < 0$ and $\frac{\partial \varphi_V}{\partial \nu} < 0$ on $\partial\Omega$. It follows that $\frac{\varphi_V^q}{u^{q-1}}$ and $\frac{\varphi_V^{q-p+1}}{u^{q-p}}$ can be tested in (P) and the eigenvalue problem, respectively. Thus we get:

$$\begin{aligned} \int f|\varphi_V|^q &= q \int \left(\frac{\varphi_V}{u}\right)^{q-1} |\nabla u|^{p-2} \nabla u \nabla \varphi_V \\ &\quad - (q-1) \int \left(\frac{\varphi_V}{u}\right)^q |\nabla u|^p + \int V u^{p-q} \varphi_V^q \end{aligned} \tag{2.2}$$

and

$$\lambda_1(V) \int u^{p-q} \varphi_V^q = (q-p+1) \int \left(\frac{\varphi_V}{u}\right)^{q-p} |\nabla \varphi_V|^p \tag{2.3}$$

$$-(q - p) \int \left(\frac{\varphi_V}{u}\right)^{q-p+1} |\nabla \varphi_V|^{p-2} \nabla \varphi_V \nabla u + \int V u^{p-q} \varphi_V^q$$

On the other hand, from Picone’s identity, one has

$$\begin{aligned} &\left(\frac{\varphi_V}{u}\right)^q |\nabla u|^p - p \left(\frac{\varphi_V}{u}\right)^{q-p+1} |\nabla \varphi_V|^{p-2} \nabla \varphi_V \nabla u \\ &+ (p - 1) \left(\frac{\varphi_V}{u}\right)^{q-p} |\nabla \varphi_V|^p \geq 0 \end{aligned} \tag{2.4}$$

and also

$$\begin{aligned} &\left(\frac{\varphi_V}{u}\right)^{q-p} |\nabla \varphi_V|^p - p \left(\frac{\varphi_V}{u}\right)^{q-1} |\nabla u|^{p-2} \nabla u \nabla \varphi_V \\ &+ (p - 1) \left(\frac{\varphi_V}{u}\right)^{q-1} |\nabla u|^{p-2} \nabla u \nabla \varphi_V \geq 0. \end{aligned} \tag{2.5}$$

Adding up (4) multiplied by $\left(\frac{q}{p} - 1\right)$ and (5) multiplied by $\frac{q}{p}$ and integrating over Ω we get the same result as subtracting (2) from (3), which yields the conclusion. □

Proof of Theorem 1.2

Consider the following infima:

$$c_1 = \inf \left\{ E_V(u); \int f|u|^q = -1 \right\} \quad \text{and} \quad c_2 = \inf \left\{ E_V(u); \int f|u|^q = 1 \right\}.$$

We will show that these infima are achieved. Moreover, we will prove that $c_1 < 0$ and $c_2 > 0$, so that the associated minimizers correspond to positive solutions of (P).

Step 1: $c_1 < 0$.

Let w be defined by

$$w = \frac{\varphi_V}{\left(-\int f \varphi_V^q\right)^{\frac{1}{q}}};$$

then $\int f w^q = -1$, so that

$$c_1 \leq E_V(w) = \frac{E_V(\varphi_V)}{\left(-\int f \varphi_V^q\right)^{\frac{p}{q}}} = \frac{\lambda_1(V)}{\left(-\int f \varphi_V^q\right)^{\frac{p}{q}}} < 0.$$

Step 2: c_1 is well defined, i.e., $c_1 > -\infty$.

Assume by contradiction that there is a sequence of nonnegative function u_n such that $\int f u_n^q = -1$ and

$$E_V(u_n) = \int |\nabla u_n|^p + \int V^+ u_n^p - \int V^- u_n^p \rightarrow -\infty.$$

It follows that $\int V^- u_n^p \rightarrow \infty$. Define w_n by

$$w_n = \frac{u_n}{\left(\int V^- u_n^p\right)^{\frac{1}{p}}},$$

so that $\int f w_n^q \rightarrow 0$ and $\int V^- w_n^p = 1$. Moreover, for n large,

$$\int |\nabla w_n|^p + \int V^+ w_n^p - 1 \leq 0,$$

thus $\|w_n\|$ is bounded. We can assume that $w_n \rightharpoonup w$ in $W_0^{1,p}(\Omega)$, and so $\int f w^q = 0$, $\int V^- w^p = 1$ and $E_V(w) \leq 0$. Define $w_0 = \frac{w}{\|w\|_p}$, then

$$E_V(w_0) \leq 0, \quad \int f w_0^q = 0, \quad \text{and} \quad \|w_0\|_p = 1.$$

This is in contradiction with $\alpha(V) > 0$.

Step 3: c_1 is achieved.

Let u_n be a nonnegative minimizing sequence, i.e. $\int f u_n^q = -1$ and $E_V(u_n) \rightarrow c_1$. We claim that $\|u_n\|$ is bounded. If not, from

$$E_V(u_n) = \int |\nabla u_n|^p + \int V^+ u_n^p - \int V^- u_n^p \rightarrow c_1,$$

we have that $\int V^- u_n^p \rightarrow \infty$. Again, define

$$w_n = \frac{u_n}{\left(\int V^- u_n^p\right)^{\frac{1}{p}}},$$

so that $\|w_n\|$ is bounded. We can assume that $w_n \rightharpoonup w$ in W_0^1 , and so $\int f w^q = 0$, $\int V^- w^2 = 1$ and $E_V(w) = 0$. As in Step 2, we have a contradiction with the assumption $\alpha(V) > 0$.

Now, we can assume that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, so that $\int f |u|^q = -1$ and

$$c_1 \leq E_V(u) \leq \liminf E_V(u_n) = c_1.$$

Thus u is a minimizer for c_1 .

Step 4 c_2 is well defined, is achieved and is positive.

The first two assertions can be proved as for c_1 . In order to prove that $c_2 > 0$, let u be a minimizer for c_2 . Then, up to a multiplicative constant, u is a solution of

$$-\Delta_p u + V(x)u^{p-1} = c_2 f(x)u^q, \quad u \in W_0^{1,p}(\Omega).$$

From Lemma 2, there holds

$$c_2 \int f \varphi_V^q \leq \lambda_1(V) \int \varphi_V^q u^{p-q} < 0.$$

Thus $c_2 > 0$. This finishes the proof of Theorem 1.2. □

Next we prove Corollary 1, as a consequence of the next lemma, which shows that the condition $\alpha(V) > 0$ can be obtained if we control the sign of $\lambda_1(V)$ and $\int f \varphi_V^q$ in an appropriate way:

Lemma 2.3. *Given $R, \delta > 0$ there is $\varepsilon_0 > 0$ such that for all V with $\|V\|_r \leq R$, $\int f \varphi_V^q < -\delta$ and $\lambda_1(V) > -\varepsilon_0$, we have $\alpha(V) > 0$.*

Proof. Let ε_0 be given by the previous lemma. We have that $\lambda_1(V - \lambda_1(V)) = 0$ and it is achieved by φ_V , so that $\alpha(V - \lambda_1(V)) \geq \varepsilon_0$. Then, for $u \in W_0^{1,p}(\Omega)$, we have

$$E_{V+\lambda_1(V)}(u) \geq \varepsilon \quad \text{if } \|u\|_p = 1 \quad \text{and} \quad \int f|u|^q = 0,$$

and so

$$E_V(u) \geq \lambda_1(V) + \varepsilon \quad \text{if } \|u\|_p = 1 \quad \text{and} \quad \int f|u|^q = 0.$$

Thus $\alpha(V) \geq \lambda_1(V) + \varepsilon > 0$. □

Now we turn to the proof of Proposition 1.4:

Proof of Proposition 1.4. We set $F(u) = \int_{\Omega} f|u|^q$ and $G(u) = \|u\|_p^p$ for $u \in W_0^{1,p}(\Omega)$. Let us show that $F'(u_0)$ and $G'(u_0)$ are linearly independent if u_0 realizes $\tilde{\alpha}(V)$. Assume that $a\langle F'(u_0), v \rangle + b\langle G'(u_0), v \rangle = 0$ for every $v \in W_0^{1,p}(\Omega)$. Taking $v = u_0$ we get $b = 0$, so that $a\langle F'(u_0), v \rangle = 0$ for every $v \in$

$W_0^{1,p}(\Omega)$. If $a \neq 0$ then $f|u_0|^{q-1} \equiv 0$, a contradiction. Therefore one can apply Lagrange multipliers rule to infer that u_0 is a solution of $-\Delta_p u + Vu^{p-1} = t_0 f u_0^{q-1} + s_0 u_0^{p-1}$. Multiplying this equation by u_0 we get $s_0 = \alpha(V) = 0$. Furthermore, from Lemma 2, we have

$$t_0 \int f \varphi_V^q \leq \lambda_1(V) \int \varphi_V^q u^{p-q} < 0,$$

so that $t_0 > 0$. Therefore, after rescaling, u_0 provides a solution for $P_{V,f}$. □

3 Nonexistence of solutions

3.1 Proof of Theorem 1.5

Repeating the proof of Lemma 2 we obtain

$$0 = q \int \left(\frac{\varphi_V}{u}\right)^{q-1} |\nabla u|^{p-2} \nabla u \nabla \varphi_V - \tag{3.1}$$

$$(q-1) \int \left(\frac{\varphi_V}{u}\right)^q |\nabla u|^p + \int Vu^{p-q} \varphi_V^q$$

and

$$0 = (q-p+1) \int \left(\frac{\varphi_V}{u}\right)^{q-p} |\nabla \varphi_V|^p - \tag{3.2}$$

$$(q-p) \int \left(\frac{\varphi_V}{u}\right)^{q-p+1} |\nabla \varphi_V|^{p-2} \nabla \varphi_V \nabla u + \int Vu^{p-q} \varphi_V^q,$$

so that

$$\left(\frac{\varphi_V}{u}\right)^q |\nabla u|^p - p \left(\frac{\varphi_V}{u}\right)^{q-p+1} |\nabla \varphi_V|^{p-2} \nabla \varphi_V \nabla u + \tag{3.3}$$

$$(p-1) \left(\frac{\varphi_V}{u}\right)^{q-p} |\nabla \varphi_V|^p \equiv 0$$

and

$$\left(\frac{\varphi_V}{u}\right)^{q-p} |\nabla \varphi_V|^p - p \left(\frac{\varphi_V}{u}\right)^{q-1} |\nabla u|^{p-2} \nabla u \nabla \varphi_V + \tag{3.4}$$

$$(p-1) \left(\frac{\varphi_V}{u}\right)^{q-1} |\nabla u|^{p-2} \nabla u \nabla \varphi_V \equiv 0.$$

Therefore Picone's identity states that u and φ_V are proportional, so that, up to a multiplicative constant, one has $f \equiv \frac{\lambda_1(V)}{\varphi_V^{q-p}}$, which is impossible. □

3.2 Proof of Theorem 1.6

Assume the existence of $R_0 > 0$ and a sequence $V_n \in B$ such that $\lambda_1(V_n) \leq -R_0$, $\int f|\varphi_n|^q \geq -\frac{1}{n}$ (where $\varphi_n = \varphi_{V_n}$) and that, for $V = V_n$, (P) has a solution u_n . We may assume that $V_n \rightharpoonup V_0$ in $L^\infty(\Omega)$ so that, by weak continuity, $\lambda_1(V_0) \leq -R_0$ and $\int f|\varphi_0|^q \geq 0$. Let us now show that u_n converges to a solution of (P), for $V = V_0$. Lemma 2 will then provide a contradiction. From [3, Lemma 3.2] we have that the sequence u_n is bounded in $L^\infty(\Omega)$ and therefore, by (2.1), it is bounded in $W_0^{1,p}(\Omega)$. Hence $u_n \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$, strongly in $L^{q_{s'}}(\Omega)$. Multiplying (P), with $V = V_n$, by $u_n - u_0$ one gets that $\langle E'_{V_n}(u_n), u_n - u_0 \rangle \rightarrow 0$ so that, by the (S^+) property of the p-laplacian, $u_n \rightarrow u_0$ in $W_0^{1,p}(\Omega)$. Therefore u_0 is a solution of (P) for $V = V_0$. \square

4 The parameter dependent problem

Let $\lambda > 0$ a parameter and consider the problem

$$(P_\lambda) \quad \begin{cases} -\Delta_p u = \lambda u^{p-1} + f(x)u^{q-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where we assume that $1 < p < q < p^*$. Here f is a bounded function which changes sign in Ω . More precisely, we will assume that $\Omega^+ = \{x \in \Omega : f(x) > 0\}$ and $\Omega^- = \{x \in \Omega : f(x) < 0\}$ are open and nonempty sets.

The solutions of (P_λ) are the critical points of the following C^1 -functional

$$I(u) = \frac{1}{p} \int_\Omega |\nabla u|^p - \frac{\lambda}{p} \int_\Omega (u^+)^p - \frac{1}{q} \int_\Omega f(x)(u^+)^q, \quad u \in W_0^{1,p}(\Omega).$$

Define

$$\Lambda = \{\lambda : (P_\lambda) \text{ has a nontrivial solution}\} \quad \text{and} \quad \lambda^* = \sup \Lambda.$$

It follows from Theorem 1.2 that $\Lambda \neq \emptyset$. In fact $(-\infty, \lambda_1) \subset \Lambda$. Here λ_1 denotes the first eigenvalue of

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, it is a consequence of Corollary 1.3 that if $\int f\varphi_1^q < 0$, where $\varphi_1 > 0$ is the L^p -normalized eigenfunction associated to λ_1 , then $\lambda^* > \lambda_1$. Also,

we know that $\lambda^* \leq \lambda_1(\Omega \setminus \overline{\Omega^-})$, see the begin of the proof of Lemma 4.2 in Appendix.

4.1 Proof of Theorem 1.7

The proof will be divided in several steps.

Step 1. There is a nontrivial solution for all $\lambda < \lambda^*$.

Pick $\lambda < \lambda^*$ and $\bar{\lambda} > \lambda$ with $\bar{\lambda} \in \Lambda$. Let \bar{u} be a nontrivial solution of $(P_{\bar{\lambda}})$. Then

$$-\Delta_p \bar{u} = \bar{\lambda} \bar{u}^{p-1} + f(x) \bar{u}^{q-1} \geq \lambda \bar{u}^{p-1} + f(x) \bar{u}^{q-1},$$

so that \bar{u} is a supersolution of (P_λ) . Now consider

$$M = \{u \in W_0^{1,p}(\Omega); 0 \leq u \leq \bar{u}\}.$$

Let $u_1 \in M$ such that $I(u_1) = \inf_M I$, then u_1 is a solution of (P_λ) , see for instance [16, Theorem I.2.4]. In order to prove that u_1 is nontrivial, let $0 \leq \phi \in C_0^1(\Omega)$ with support compact and such that

$$\frac{\int_\Omega |\nabla \phi|^p}{\int_\Omega \phi^p} < \lambda. \tag{4.1}$$

The existence of ϕ holds since $\lambda_1 < \lambda$. There is $s_0 > 0$ such that $s\phi \leq \bar{u}$ for $0 < s < s_0$. Moreover, by (4.1) we have

$$I(s\phi) \leq \frac{s^p}{p} \left(\int_\Omega |\nabla \phi|^p - \lambda \int_\Omega \phi^p \right) - \frac{s^q}{q} \int_\Omega f(x) \phi^q < 0$$

if $s > 0$ is small enough (recall that $p < q$). Then $I(u_1) < 0$, and so u_1 is a nontrivial solution.

Step 2. Let u_1 be the solution of (P_λ) constructed in Step 1. We claim that u_1 can be assumed to be a local minimizer of I in $W_0^{1,p}(\Omega)$.

Case 1. Assume $\partial\Omega$ is C^2 . By classical regularity results, $u \in C_0^1(\overline{\Omega})$. Then, by the strong maximum principle, we have

$$u_1 > 0 \text{ in } \Omega \quad \text{and} \quad \partial_\nu u_1 < 0 \text{ on } \partial\Omega,$$

where ∂_ν denotes the normal exterior derivative. Moreover, we have that $u_1 \leq \bar{u}$, where \bar{u} is a nontrivial solution of $(P_{\bar{\lambda}})$ with $\lambda < \bar{\lambda} < \lambda^*$. Applying the comparison principle, see [8, Proposition 3.4], we can conclude that

$$\bar{u} > u_1 > 0 \text{ in } \Omega \quad \text{and} \quad \partial_\nu \bar{u} < \partial_\nu u_1 < 0 \text{ on } \partial\Omega.$$

That means that M contains a $C_0^1(\Omega)$ neighborhood of u_1 and so u_1 is a local minimum in C_0^1 . By classical results, we can conclude that u_1 is a local minimum in $W_0^{1,p}(\Omega)$.

Case 2. Assume that $p \geq 2$. Set \bar{u} and $\bar{\lambda}$ as before. We have that $u_1 \leq \bar{u}$ a.e. in Ω . Set $\lambda' = (\lambda + \bar{\lambda})/2$ and let $\delta > 1$ such that

$$(\delta^{q-p} - 1) b(x) \bar{u}^{q-p} < \bar{\lambda} - \lambda'.$$

Hence, multiplying the above inequality by $\delta^{p-1} \bar{u}^{p-1}$, we obtain

$$-\Delta_p(\delta \bar{u}) = \delta^{p-1} \bar{\lambda} \bar{u}^{p-1} + \delta^{p-1} f(x) \bar{u}^{q-1} \geq \delta^{p-1} \lambda' \bar{u}^{p-1} + \delta^{q-1} f(x) \bar{u}^{q-1}.$$

Thus $\delta \bar{u}$ is a supersolution for (P_λ) and moreover $u_1 < \delta \bar{u}$ a.e. in Ω . Without loss of generality, we can assume that

$$I(u_1) = \min\{F(u) : 0 \leq u(x) \leq \delta \bar{u}(x) \text{ a.e. } x \in \Omega\}.$$

It follows by Lemma 4.1 (see Appendix) that u_1 can be assumed to be a local minimizer of I in $W_0^{1,p}$.

Step 3. We will apply the mountain pass theorem to obtain the second nontrivial solution to (P_λ) .

Consider now the functional

$$J(u) = \frac{1}{p} \int_\Omega |\nabla(u_1 + u)|^p - \int_\Omega l(x, u) - \int_\Omega G(x, u),$$

where

$$l(x, s) = \frac{\lambda}{p} ((u_1 + s)^+)^p - \frac{\lambda}{p} u_1^p.$$

and

$$G(x, s) = \frac{1}{q} f(x) ((u_1 + s)^+)^q - \frac{1}{q} f(x) u_1^q.$$

It is clear that if u is a nontrivial critical point of J , then $u_1 + u \geq 0$, so that $u_1 + u$ is a solution of (P_λ) different from u_1 .

Since u_1 is a local minimizer of I and $J(u) = I(u_1 + u) - I(u_1) + \|u_1\|^2$, it follows that there is $r > 0$ such that

$$J(u) \geq J(0) \quad \text{for all } u \in W_0^{1,p}(\Omega) \text{ with } \|u\| \leq r.$$

Now, fix $0 \leq v \in W_0^{1,p}(\Omega)$ such that

$$\int_{\Omega} f(x)v^q > 0.$$

Then

$$\begin{aligned} J(sv) &= \frac{s^p}{p} \int_{\Omega} |\nabla(v + \frac{u_1}{s})|^p - \frac{s^p \lambda}{p} \int_{\Omega} m(x)(v + \frac{u_1}{s})^p \\ &+ \frac{\lambda}{p} \int_{\Omega} m(x)u_1^p - \frac{s^q}{q} \int_{\Omega} f(x)(v + \frac{u_1}{s})^q + \frac{1}{q} \int_{\Omega} f(x)u_1^q \\ &\rightarrow -\infty \text{ as } s \rightarrow \infty. \end{aligned}$$

Finally, if u_n is a $(PS)_c$ sequence for J , then it follows that $u_1 + u_n$ is a $(PS)_{c+I(u_1)-\|u_1\|^2}$ for I . Thus J satisfies the (PS) condition since I satisfies the (PS) condition if $\lambda < \lambda^*$, see Lemma 4.2 in Appendix.

□

4.2 Appendix

Here we will denote by $\bar{w} := \delta \bar{u}$.

Lemma 4.1. *Assume that $p \geq 2$. Suppose that u_1 is the unique minimizer of I restricted to $M = \{u \in W_0^{1,p}(\Omega) : 0 \leq u(x) \leq \bar{w}(x) \text{ a.e. } x \in \Omega\}$. Then u_1 is a local minimizer of I_{λ} in $W_0^{1,p}(\Omega)$.*

Proof. Consider the set

$$M_n = \left\{ u \in W_0^{1,p}(\Omega) : \text{dist}(u, M) \leq \frac{1}{n} \right\}.$$

It is easy to verify that M_n is weakly closed, and I is coercive and weakly lower semi-continuous on M_n with respect to the norm of $W_0^{1,p}$. Then, by [16, Theorem I.1.2], there is $u_n \in M_n$ such that

$$I(u_n) = \min_{M_n} I.$$

It follows that

$$I'(u_n) \cdot (u_n - \bar{w})^+ \leq 0,$$

i.e.,

$$\begin{aligned} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - \bar{w})^+ dx &\leq \lambda \int_{\Omega} u_n^{p-1} (u_n - \bar{w})^+ dx \\ &+ \int_{\Omega} f(x) u_n^{q-1} (u_n - \bar{w})^+ dx. \end{aligned}$$

Moreover, we have that

$$\begin{aligned} \int_{\Omega} |\nabla \bar{w}|^{p-2} \nabla \bar{w} \nabla (u_n - \bar{w})^+ dx &\geq \lambda' \int_{\Omega} \bar{w}^{p-1} (u_n - \bar{w})^+ dx \\ &+ \int_{\Omega} f(x) \bar{w}^{q-1} (u_n - \bar{w})^+ dx. \end{aligned}$$

Now, fix $\epsilon > 0$ such that

$$\lambda' \geq (1 + \epsilon)\lambda + \epsilon f(x) \bar{w}^{q-p}.$$

Multiplying this inequality by \bar{w}^{p-1} we get

$$\lambda' \bar{w}^{p-1} \geq (1 + \epsilon)\lambda \bar{w}^{p-1} + \epsilon f(x) (\bar{w})^{q-1}.$$

It follows that

$$\begin{aligned} \int_{\Omega} |\nabla \bar{w}|^{p-2} \nabla \bar{w} \nabla (u_n - \bar{w})^+ dx &\geq \lambda \int_{\Omega} (1 + \epsilon) \bar{w}^{p-1} (u_n - \bar{w})^+ dx \\ &+ \int_{\Omega} f(x) (1 + \epsilon) \bar{w}^{q-1} (u_n - \bar{w})^+ dx. \end{aligned}$$

Remark that for $p \geq 2$ the p -Laplacian has the strongly monotone property, so we can conclude that

$$\int_{\Omega} |\nabla (u_n - \bar{w})^+|^p dx \leq \int_{\Omega} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla \bar{w}|^{p-2} \nabla \bar{w}) \nabla (u_n - \bar{w})^+ dx.$$

Using the above inequalities, we get that

$$\begin{aligned} \int_{\Omega} |\nabla (u_n - \bar{w})^+|^p dx &\leq \lambda \int_{\Omega} (u_n^{p-1} - (1 + \epsilon) \bar{w}^{p-1}) (u_n - \bar{w})^+ dx \\ &+ \int_{\Omega} f(x) (u_n^{q-1} - (1 + \epsilon) \bar{w}^{q-1}) (u_n - \bar{w})^+ dx. \end{aligned}$$

There is a constant $C = C(\epsilon) > 0$ such that, for $a > b \geq 0$,

$$a^{p-1} - b^{p-1} \leq \epsilon b^{p-1} + C(a - b)^{p-1}$$

and

$$a^{q-1} - b^{q-1} \leq \epsilon b^{q-1} + C(a - b)^{q-1}.$$

Therefore

$$\begin{aligned} \|(u_n - \bar{w})^+\|^p &\leq \lambda C \int_{\Omega} [(u_n - \bar{w})^+]^p dx \\ &\quad + C \int_{\Omega} f(x) [(u_n - \bar{w})^+]^q dx \\ &\leq C |\{x : u_n(x) > \bar{w}(x)\}|^{\frac{p}{N}} \|(u_n - \bar{w})^+\|^p \\ &\quad + C \|(u_n - \bar{w})^+\|^{q-p} \|(u_n - \bar{w})^+\|^p. \end{aligned}$$

Note that u_n converges to a minimizer of I in M , so from the uniqueness of u_0 it follows that $\|u_n - u_0\| \rightarrow 0$ as $n \rightarrow \infty$. Moreover, since $u_0 < \bar{w}$, it follows that $|\{x : u_n(x) > \bar{w}(x)\}| \rightarrow 0$ as $n \rightarrow \infty$. Thus

$$\|(u_n - \bar{w})^+\|^p \leq o(1) \|(u_n - \bar{w})^+\|^p,$$

so that there is n_0 such that $(u_n - \bar{w})^+ = 0$ for $n \geq n_0$, and so $u_n \leq \bar{w}$. As a consequence, we have that $u_n^+ \in M$ and so $I(u_n^+) \geq I(u_1)$. Now, if $n \geq n_0$, then

$$I(u_1) \geq I(u_n) = \frac{1}{p} \int_{\Omega} |\nabla u_n^-|^p dx + I(u_n^+) \geq \frac{1}{p} \int_{\Omega} |\nabla u_n^-|^p dx + I(u_1).$$

We infer that $u_n^- = 0$, and so $u_n \in M$ for $n \geq n_0$. Therefore u_1 is a local minimizer of I in $W_0^{1,p}$.

□

Lemma 4.2. *The functional I_{λ} satisfies the (PS) condition if $\lambda < \lambda^*$.*

Proof. First, note that $\lambda^* \leq \lambda_1(\Omega^*)$, where $\Omega^* = \Omega \setminus \overline{\Omega^-}$. Actually, let u be a solution of (P_{λ}) and ψ be the first eigenfunction associated to $\lambda_1(\Omega^*)$. By Picone's identity (c.f. [2]),

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla (\phi^p / u^{p-1}) dx \leq \int_{\Omega} |\nabla \phi|^p dx,$$

It follows that, using the equation (P_{λ}) ,

$$\lambda \int_{\Omega} \phi^p + \int_{\Omega} f^+(x) u^{p-q} \phi^p \leq \lambda_1(\Omega^*) \int_{\Omega} \phi^p,$$

and consequently $\lambda \leq \lambda_1(\Omega^*)$. Thus $\lambda_1(\Omega^*)$ is an upper bound for Λ , and so $\lambda^* \leq \lambda_1(\Omega^*)$.

Now, consider a sequence $u_n \in W_0^{1,p}$ such that

$$I(u_n) \leq c \quad \text{and} \quad I'(u_n) \rightarrow 0.$$

We need to show that u_n has a convergent subsequence. But since $q < p^* - 1$, it is enough to show that u_n is bounded in $W_0^{1,p}(\Omega)$. Note that

$$I(u_n) - \frac{1}{q}I'(u_n)u_n = \left(\frac{1}{p} - \frac{1}{q}\right) (\|u_n\|^p - \lambda\|u_n^+\|_p^p) \leq c + c\|u_n\|, \tag{4.2}$$

so that the lemma follows from the next claim:

Claim.: The sequence u_n is bounded in $L^p(\Omega)$.

Suppose by contradiction that $\|u_n\|_p \rightarrow \infty$ as $n \rightarrow \infty$. Set $v_n = u_n/\|u_n\|_p$. From (4.2), we have that v_n is bounded. So we can assume that $v_n \rightharpoonup v_0$ in $W_0^{1,p}$ with $\|v_0\|_p = 1$. For any $w \in W_0^{1,p}$ we know that $I'(u_n)w \rightarrow 0$ as $n \rightarrow \infty$. In particular

$$\begin{aligned} \|u_n\|_2^{q-p} \int_{\Omega} f(x)(v_n^+)^{q-1}w dx &= \int_{\Omega} [|\nabla v_n|^{p-2}\nabla v_n \nabla w - \lambda(v_n^+)^{p-1}w] dx \tag{4.3} \\ &+ o(1). \end{aligned}$$

Consequently

$$\int_{\Omega} f(x)(v_0^+)^{q-1}w dx = 0 \text{ for all } w \in W_0^{1,p}(\Omega). \tag{4.4}$$

If $\Omega_0 := \Omega \setminus (\overline{\Omega^+ \cup \Omega^-})$ is empty, then (4.4) implies that $v_0 = 0$, but it is a contradiction since $\|v_0\|_p = 1$.

On the other hand, if $\Omega_0 \neq \emptyset$, then (4.4) implies that $v_0 \in W_0^{1,p}(\Omega_0)$. Now, passing a subsequence, we can assume that

$$\int_{\Omega} |\nabla v_n|^{p-2}\nabla v_n \nabla w \rightarrow \int_{\Omega} |\nabla v_0|^{p-2}\nabla v_0 \nabla w,$$

see for instance [11, 18]. From (4.3) we have that

$$\int_{\Omega} [|\nabla v_0|^{p-2}\nabla v_0 \nabla w - \lambda(v_0^+)^{p-1}w] dx = 0 \text{ for all } w \in W_0^{1,p}(\Omega_0).$$

It follows that $-\Delta_p v_0 = \lambda(v_0^+)^{p-1}$, and so $v_0 \geq 0$ and $\lambda = \lambda_1(\Omega_0)$ since $\|v_0\|_p = 1$. It is a contradiction with the assumption $\lambda < \lambda^*$, since $\lambda^* \leq \lambda_1(\Omega^*) < \lambda_1(\Omega_0)$.

□

References

- [1] S. Alama & G. Tarantello, *On semilinear elliptic equations with indefinite nonlinearities*. Calc. Var. Partial Differential Equations **1** (1993), 439-475.
- [2] W. Allegretto & Y. X. Huang, *A Picone's identity for the p -Laplacian and applications*. Nonl. Anal. **32** (1998), 819-830.
- [3] T. Bartsch & Z. Liu, *On a superlinear elliptic p -Laplacian equation*. J. Differential Equations **198** (2004), 149-175.
- [4] H. Berestycki, I. Capuzzo-Dolcetta & L. Nirenberg, *Variational methods for indefinite superlinear homogeneous elliptic problems*. NoDEA Nonlinear Differential Equations Appl. **2** (1995), 553-572.
- [5] I. Birindelli & F. Demengel, *Existence of solutions for semilinear equations involving the p -Laplacian: the non coercive case*, Calc. Var. Partial Differential Equations **20** (2004), 343-366.
- [6] J. Chabrowski & J. Marcos do , *On some fourth-order semilinear elliptic problems in \mathbb{R}^n* . Nonlinear Anal. **49** (2002), 861884.
- [7] M. Cuesta & H. Ramos Quoirin, *A weighted eigenvalue problem for the p -laplacian plus a potential*, NoDEA Nonlinear Differential Equations Appl. **16** (2009), 469-491.
- [8] D.G. de Figueiredo, J.-P. Gossez & P. Ubilla, *Local superlinearity and sublinearity for the p -Laplacian*, J. Func. Anal. **257** (2009), 721-752
- [9] M. Del Pino, *Positive solutions of a semilinear elliptic equation on a compact manifold*, Nonlin. Anal. 22 (11) (1994) 1423-1430.
- [10] P. Drbek, Y.X. Huang, *Xi Multiple positive solutions of quasilinear elliptic equations in R^N* . Nonlinear Anal. 37 (1999), no. 4, Ser. A: Theory Methods, 457-466.
- [11] J. M. do Ó, *Semilinear dirichlet problems for the N -Laplacian in R^N with nonlinearities in the critical growth range*. Differential and Integral Equations **9** (1996), 967-979.
- [12] Y. Il'yasov, *On positive solutions of indefinite elliptic equations*. C. R. Acad. Sci. Paris Sr. I Math. **333** (2001), 533-538.
- [13] T. Ouyang, *On the positive solutions of semilinear equations $\Delta u + \lambda u - hu^p = 0$ on the compact manifolds*. Trans. Amer. Math. Soc. **331** (1992), 503-527.
- [14] H. Tehrani, *On indefinite superlinear elliptic equations*. Calc. Var. Partial Differential Equations **4** (1996), 139-153
- [15] J.-L. Lions, *it Équations différentielles opérationnelles et problèmes aux limites*, Die Grundlehren der mathematischen Wissenschaften, Bd. 111 Springer-Verlag, Berlin-Gttingen-Heidelberg 1961 J.L. Lions, *E* , Birkhäuser, Boston (1993).

- [16] M. Struwe, *Variational Methods, Applications to Nonlinear PDE and Hamiltonian Systems*, Springer-Verlag, Berlin, 1996.
- [17] J.L Vazquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. 12 (1984), no. 3, 191–202.
- [18] J. Yang & X. P. Zhu, *On the existence of nontrivial solution of a quasilinear elliptic boundary value problem for unbounded domains. II. Zero mass case*. Acta Math. Sci. (English Ed.) **7** (1987), 341-359.

Universidade Federal de São Carlos,
Departamento de Matemática, Caixa Postal 676,
CEP: 13565-905, São Carlos - SP, BRAZIL
e-mail: odair@dm.ufscar.br

ULB, CP 214 Bd du Triomphe B-1050
Bruxelles BELGIUM
e-mail: hramosqu@ulb.ac.be