Matemática Contemporânea, Vol 40, 91-102

Immersion of almost Ricci solitons into a Riemannian manifold

Abdênago Barros[1], **José N. Gomes**[2] [*]
and
Ernani Ribeiro Jr.[3] [†]

Dedicated to Professor Gervásio Colares on the occasion of his 80th birthday.

Abstract

The principal aim of this short paper is to study immersions of an almost Ricci soliton or a Ricci soliton (M^n, g, X, λ) into a Riemannian manifold $\widetilde{M}^{n+p}$. First we shall present some obstruction results in order to obtain a minimal immersion under conditions on the sectional curvature of $\widetilde{M}^{n+p}$. When $\widetilde{M}^{n+p}$ is a space form $\widetilde{M}^{n+p}_c$ of sectional curvature c, the pinching $\lambda \geq (n-1)(c+H^2)$ gives that such an immersion is umbilical. Finally, concerning to Ricci solitons we shall show that a shrinking Ricci soliton immersed into a space form with constant mean curvature must be the Gaussian soliton or its traceless tensor associated to the second fundamental form has supremum strictly positive.

1 Introduction and statement of the main results

Ricci solitons play a remarkable role in the study of the Ricci flow. Among their properties we detach that they are stationary points of the Ricci flow in the space of metrics on M^n modulo diffeomorphisms and scalings of M^n. It usually serves as a dilation limit of solutions to the Ricci flow. Therefore, it is very important to classify Ricci solitons or to understand their geometry. In addition, when M^n is compact, Perelman [18] reduced the study of such manifold to gradient case of a smooth function f on M^n called Perelman's potential. On the other hand, in [19] Pigola et al. modified the definition of

2000 *AMS Subject Classification.* Primary 53C25, 53C20, 53C21; Secondary 53C6
Key Words and Phrases: Ricci Soliton, almost Ricci soliton, immersion, mean curvature.
[*]Partially supported by FAPEAM
[†1,3] Both partially supported by CNPq-BR

a gradient Ricci soliton by adding the condition on the parameter λ to be a variable function. In [4], the following general definition of an almost Ricci soliton was considered.

Definition 1.1. *An almost Ricci soliton is a Riemannian manifold M^n endowed with a metric* g*, a vector field X and a soliton function $\lambda : M^n \to \mathbb{R}$ satisfying*

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1.1}$$

where $\mathcal{L}_X g$ stands for the Lie derivative of the metric g in the direction of X.

When X is a gradient vector field of a smooth function $f : M^n \to \mathbb{R}$ this definition agrees with that one given in [19]. In this case the previous equation turns out

$$Ric + \nabla^2 f = \lambda g, \tag{1.2}$$

where $\nabla^2 f$ stands for the Hessian of f.

Following the terminology of Ricci solitons, an almost Ricci soliton will be *expanding*, *steady* or *shrinking* if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively. If λ has no constant sign it will be called *indefinite*.

We point out that if λ is constant, equation (1.1) reduces to that associated to a Ricci soliton. Under this point of view an almost Ricci soliton generalizes a Ricci soliton. Moreover, when either the vector field X is trivial, or the potential f is constant, an almost Ricci soliton will be called *trivial*, while for a nontrivial almost Ricci soliton its associated potential vector field X or its function are not trivial. We notice that when $n \geq 3$ and X is a Killing vector field an almost Ricci soliton will be a Ricci soliton, since in this case we have an Einstein manifold, which implies λ constant. Therefore, the study of almost Ricci solitons will be interesting when the field X is not a Killing field.

Taking into account Perelman's potential for Ricci soliton it was proved in [2], that up to constant, this potential is the function which appears on the Hodge-de Rham decomposition associated to the 1-form $X^\flat$. In the noncompact case, there exist non gradient Ricci solitons, see Bair and Danielo [3] and Lott [15]. In [17] it was proved that every noncompact shrinking soliton is a gradient soliton. We point out that an ancient solution of the Ricci flow has nonnegative scalar curvature, see [10]. For more details about Ricci solitons we recommend [8]. Moreover, in order to complete our ingredients we also recall that the Gaussian soliton is the Euclidean space $\mathbb{R}^n$ endowed with its standard metric $||\,||$ and the potential $f(x) = \frac{\lambda}{2}||x||^2$.

Into the direction of understand the geometry of almost Ricci solitons it was proved in [19] some conditions to existence of gradient almost Ricci solitons. Moreover, in [4] it was proved some structural equations and rigidity theorems to almost Ricci solitons. For more details about the geometry of almost Ricci solitons see [4] and [19]. For a locally conformally flat gradient almost Ricci soliton, Catino proved in [9], that around any regular point of f, such manifold is locally a warped product with $(n-1)$-dimensional fibers of constant sectional curvature. Recently, in [16] it was proved that under some analytic conditions a steady or shrinking Ricci soliton minimally immersed into a Euclidean space is totally geodesic.

In order to proceed we remember a result due to Yau [21] which is a generalization of Hopf's maximum principle: a subharmonic function $f : M^n \to \mathbb{R}$ defined over a complete noncompact Riemannian manifold is constant, provided its gradient belongs to $\mathrm{L}^1(M^n)$. Recently this result was extended by Camargo et al. [7] for a vector field X. With the aid of this extension we derive our first result. This result will give conditions for nonexistence of minimal immersion of an almost Ricci soliton into a Riemannian manifold. More precisely, we have the following theorem.

Theorem 1.2. *Let $\varphi : M^n \looparrowright \widetilde{M}^{n+p}$ be an isometric immersion of an almost Ricci soliton $(M^n,\, g\,, X,\, \lambda)$ into a Riemannian manifold $\widetilde{M}^{n+p}$ of sectional curvature $\widetilde{k}$. Then the following conditions hold.*

1. *If $|X| \in \mathcal{L}^1(M)$, $\widetilde{k} \leq 0$ and $\lambda > 0$, then φ can not be minimal.*

2. *If $|X| \in \mathcal{L}^1(M)$, $\widetilde{k} < 0$ and $\lambda \geq 0$, then φ can not be minimal.*

3. *If $|X| \in \mathcal{L}^1(M)$, $\widetilde{k} \leq 0$, $\lambda \geq 0$ and φ is minimal, then M^n is flat and totally geodesic.*

4. *If $\sup_M |X| < \infty$, $\widetilde{k} \leq 0$ and $\lambda \geq c > 0$, where $c \in \mathbb{R}$, then φ can not be minimal.*

As a consequence of Theorem 1.2 we shall show a condition for nonexistence of minimal immersion of shrinking Ricci soliton into a Riemannian manifold of non positive sectional curvature. More precisely, we derive the following corollary.

Corollary 1.3. *Let $\varphi : M^n \looparrowright \widetilde{M}^{n+p}$ be an isometric immersion of a shrinking Ricci soliton (M^n, g, X, λ) into a Riemannian manifold $\widetilde{M}^{n+p}$ of sectional curvature $\widetilde{k} \leq 0$. If $|X| \in \mathcal{L}^1(M)$, then φ can not be minimal.*

Now we recall that if $\sup_M |X| < \infty$ and $(M^n,\, g,\, X\,, \lambda)$ is a shrinking Ricci soliton, Theorem 1 of [22] gives that M is compact. Therefore, we have the following corollary.

Corollary 1.4. *Let $\varphi : M^n \looparrowright \mathbb{M}_c^{n+p}$ be an isometric immersion of a shrinking Ricci soliton $(M^n,\, g,\, X\,, \lambda)$ into a space form $\mathbb{M}_c^{n+p}$ of sectional curvature c. If $\sup_M |X| < \infty$ and $c \leq 0$, then φ can not be minimal.*

One notices that when M^n is compact the assumption of $|X| \in \mathcal{L}^1(M)$ is clearly satisfied in Theorem 1.2. Moreover, under compactness assumption M^n can not be simply connected, since by Kuiper [13], it will be conformal to a Euclidean sphere, which gives a contradiction with flatness.

Now we shall consider an almost Ricci soliton immersed into a Riemannian manifold of constant sectional curvature to obtain the following result.

Theorem 1.5. *Let (M^n, g, X, λ) be an almost Ricci soliton immersed into a Riemannian manifold $\widetilde{M}_c^{n+p}$ of constant sectional curvature c. Then we get:*

1. *If $|X| \in \mathcal{L}^1(M)$ and $\lambda \geq (n-1)c + n|H|^2$, then (M^n, g) is totally geodesic, with $\lambda = (n-1)c$ and scalar curvature $R = n(n-1)c$.*
2. *If M^n is compact and $\lambda \geq (n-1)c + n|H|^2$, then M^n is isometric to a Euclidean sphere.*
3. *If $|X| \in \mathcal{L}^1(M)$, $p = 1$ and $\lambda \geq (n-1)(c + H^2)$, then M^n is totally umbilical. In particular, the scalar curvature $R = n(n-1)k$ is constant, where $k = \frac{\lambda}{n-1}$ is the sectional curvature of (M^n, g).*

In [4] it was proved that a non trivial compact almost Ricci soliton is isometric to a Euclidean sphere $\mathbb{S}^n$ provided (M^n, g) has constant scalar curvature. Using this result we will obtain the following theorem.

Theorem 1.6. *Let $(M^n, g, \nabla f, \lambda)$ be a non trivial gradient compact almost Ricci soliton, minimally immersed into a unit Euclidean sphere $\mathbb{S}^{n+1}$. Suppose that $R \geq n(n-2)$, then M^n is isometric to a Euclidean sphere. Moreover, $f + \lambda$ is constant and λ satisfies the following partial differential equation:*

$$\Delta\lambda + n\lambda = n(n-1). \tag{1.3}$$

On the other hand, when M^n is a hypersurface immersed into a Riemannian space form $\mathbb{M}_c^{n+1}$ of constant sectional curvature c, it is useful to introduce the operator $\Phi = A - HI$, where A and I denote, respectively, the shape operator of the immersion and the identity operator on TM. Finally, we have the following characterization for a gradient shrinking Ricci soliton immersed into a space form.

Theorem 1.7. *Let $(M^n, g, \nabla f, \lambda)$ be a gradient shrinking Ricci soliton immersed with constant mean curvature H into a space form $\mathbb{M}_c^{n+1}$. Then we have:*

1. *either $(M^n, g, \nabla f, \lambda)$ is the Gaussian soliton, with $c \leq 0$,*
2. *or $\sup|\Phi| \geq \frac{\sqrt{n}}{2(n-1)}\{\sqrt{n^2H^2 + 4(n-1)c} - (n-2)|H|\} > 0$.*

2 Preliminaries and basic equations

In this section we shall present some preliminaries that will be used to obtain our results. First of all, we consider $M^n \looparrowright \widetilde{M}^{n+p}$ immersed as an oriented submanifold into a Riemannian manifold $\widetilde{M}^{n+p}$ and we recall Gauss equation, which is given by

$$\begin{aligned}\langle \mathrm{R}(\mathrm{X},\mathrm{Y})\mathrm{Z},\mathrm{W}\rangle =& \langle \widetilde{\mathrm{R}}(\mathrm{X},\mathrm{Y})\mathrm{Z},\mathrm{W}\rangle + \langle \alpha(\mathrm{X},\mathrm{W}),\alpha(\mathrm{Y},\mathrm{Z})\rangle \\ &- \langle \alpha(\mathrm{X},\mathrm{Z}),\alpha(\mathrm{Y},\mathrm{W})\rangle,\end{aligned} \tag{2.1}$$

where $\alpha : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)^\perp$ stands for the second fundamental form. We also recall that the mean curvature vector $H(x)$ of such an immersion at $x \in M^n$ is defined by

$$H(x) = \frac{1}{n}\sum_{i=1}^{n} \alpha(e_i, e_j),$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal frame of T_xM. In particular, taking trace of Gauss equation we obtain

$$Ric(X,Y) = c(n-1)\langle X, Y\rangle + nH\langle AX, Y\rangle - \langle AX, AY\rangle, \tag{2.2}$$

for $X, Y \in \mathfrak{X}(M)$. Hence, if $AX = HX$ we deduce

$$Ric(X,Y) = (n-1)\big(c + H^2\big)\langle X, Y\rangle. \tag{2.3}$$

Furthermore, it follows from Gauss equation that the scalar curvature R of M^n satisfies

$$R=\sum_{i,j}^{n}\langle\widetilde{R}(e_i,e_j)e_j,e_i\rangle+n^2|H|^2-\sum_{i,j}^{n}|\alpha(e_i,e_j)|^2. \tag{2.4}$$

Therefore, when $\widetilde{M}^{n+1}$ is a space form of sectional curvature c we have the next identity for the scalar curvature

$$R=n(n-1)c+n^2H^2-|A|^2. \tag{2.5}$$

In order to finish our preliminaries we recall the following results in [4].

Lemma 2.1. *Let $(M^n,\, g\,, X,\, \lambda)$ be an almost Ricci soliton. Then we have*

1. $\frac{1}{2}\Delta|X|^2=|\nabla X|^2-Ric(X,X)-(n-2)g(\nabla\lambda,X)$.
2. *If $X=\nabla f$, then*
$$\Big|\nabla^2 f-\frac{\Delta f}{n}g\Big|^2=-\frac{1}{2}\Delta R+(n-1)\Delta\lambda+\frac{R}{n}\Delta f+\frac{1}{2}\langle\nabla R,\nabla f\rangle.$$
3. *In particular, if M^n is compact and $X=\nabla f$, then*
$$\int_M\Big|\nabla^2 f-\frac{\Delta f}{n}g\Big|^2dM=\frac{(n-2)}{2n}\int_M\langle\nabla R,\nabla f\rangle dM.$$

3 Proof of the results

3.1 Proof of Theorem 1.2

Proof. If (M^n,g,X,λ), $\lambda>0$ is minimally immersed into a Riemannian manifold of sectional curvature $\widetilde{k}\leq 0$, then we conclude from equation (2.4) that $R\leq 0$. Consequently, contracting equation (1.1) we have $divX=n\lambda-R>0$, which contradicts Proposition 1 in [7], since $|X|\in\mathcal{L}^1(M)$. On the other hand, if $\widetilde{k}<0$ and $\lambda\geq 0$, then we get $divX=n\lambda-R>0$, again we derive a contradiction and this completes the proof of the two first assertions.

For the third assertion initially we notice that, under the assumptions, it follows from the previous assertions that $\widetilde{k}$, λ and R must vanish at some point $p\in M^n$, otherwise there is no such immersion. Actually, we shall show that these functions are null. To do that, pick $x\in M^n$ and let $\{e_1,\ldots,e_n\}$ be an orthonormal basis of T_xM. By using that the ambient space has sectional curvature $\widetilde{k} \leq 0$ and the immersion is minimal, we deduce from equation (2.4) that

$$R = \sum_{i,j}^{n} \widetilde{k}(e_i, e_j) - \sum_{i,j}^{n} |\alpha(e_i, e_j)|^2 \leq 0.$$

On the other hand, as $\lambda \geq 0$, we have that $divX = \lambda n - R \geq 0$. Since $|X| \in \mathcal{L}^1(M)$, we have, once more, from Proposition 1 in [7] that $divX = 0$ in M^n. Hence, we deduce $0 \geq R = \lambda n \geq 0$, i.e., $R = \lambda = 0$. This implies that, for $i, j = 1, \ldots, n$, $\widetilde{k}(e_i, e_j) = |\alpha(e_i, e_j)| = 0$ in M^n. Therefore, we conclude that M^n is totally geodesic and flat, which proves the first part of this assertion. Moreover, from Lemma 1, $\Delta|X|^2 = 2|\nabla X|^2 \geq 0$, so, if M^n is compact, we conclude by Hopf's maximum principle that $\nabla X = 0$, then (M^n, g) is Einstein, which concludes the proof of the third assertion. The assumption of the last item implies that (M^n, g) is compact and has finite first fundamental group, confront with the proof of Theorem 1.1 in [22]. Now, we assume that there exists a compact almost Ricci soliton with $\lambda \geq c > 0$ minimally immersed into a complete Riemannian manifold of non-positive sectional curvature. By using a result due to Frankel in [12] we conclude that the almost Ricci soliton must have infinite first fundamental group. Hence we obtain a contradiction and this completes the proof of the theorem. □

3.2 Proof of Theorem 1.5

Proof. Since the ambient space has constant sectional curvature equal to c, we use once more equation (2.4) to obtain

$$R = n(n-1)c + n^2|H|^2 - \sum_{i,j}^{n} |\alpha(e_i, e_j)|^2. \tag{3.1}$$

This, jointly with the hypothesis on λ, imply that

$$\begin{aligned} divX = n\lambda - R \; &= \; n\{\lambda - \big((n-1)c + n|H|^2\big)\} \\ &+ \sum_{i,j}^{n} |\alpha(e_i, e_j)|^2 \; \geq \; 0. \end{aligned} \tag{3.2}$$

Thus, we can apply again Proposition 1 in [7] to obtain $divX = 0$ in M^n. So, from equation (3.2) we conclude that M^n is totally geodesic and $\lambda = (n-1)c$. Then $H = 0$ and finally we use (3.1) to deduce $R = n(n-1)c$. If M^n is compact, as it is totally geodesic, then the ambient space is a sphere $\mathbb{S}^{n+p}$ and M^n is isometric the a Euclidean sphere $\mathbb{S}^n$, which proves the first two assertions.

For the third assertion, we can use $|A|^2 = |\Phi|^2 + nH^2$ jointly with (2.5) to infer

$$div\, X = n[\lambda - (n-1)(c + H^2)] + |\Phi|^2. \tag{3.3}$$

Hence, under the assumptions of the assertion, we can apply once more Proposition 1 in [7] to obtain $divX = 0$ on M^n. Using (3.3) we conclude $\lambda = (n-1)(c + H^2)$ and $|\Phi|^2 = 0$, which gives that M^n is totally umbilical. Thus, if we denote the principal curvatures of M^n by μ, we use Gauss equation to deduce $k - c = \mu^2$, where k is the sectional curvature of M^n. Now, a straightforward computation gives $k = c + H^2 = \frac{\lambda}{n-1}$ and $R = n\lambda = n(n-1)k$. Thus R is constant and this finishes the proof of the theorem. □

3.3 Proof of Theorem 1.6

Proof. Taking into account that the immersion is minimal we have from (2.5) that $R = n(n-1) - |A|^2$. Since we are supposing $R \geq n(n-2)$ we deduce $|A|^2 \leq n$ in M^n. Thus, it follows from Simons [20] that either $|A|^2 = 0$ or $|A|^2 = n$. Therefore, R will be constant and we can apply Corollary 1 in [4] to conclude that M^n is isometric to a standard sphere. Now we can use Chern et al. [11] or Lawson [14] to conclude that $|A|^2 = 0$. Next we use Lemma 2.1 to obtain

$$0 = \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 = (n-1)\Delta(\lambda + f),$$

which enables us to apply Hopf's maximum principle to deduce that $\lambda + f$ is constant. Moreover, we also have

$$\Delta\lambda = -\Delta f = R - n\lambda = n(n-1) - n\lambda = n(n-1-\lambda).$$

Taking into account that $\lambda = -\langle x, a\rangle + (n-1)$ is a solution of this equation, where x is the position vector of the sphere, while a is a fixed vector in $\mathbb{R}^{n+1}$, we conclude that λ is the solution of the quoted equation and we complete the proof of the theorem. □

3.4 Proof of Theorem 1.7

Proof. First we recall that a result due to [10] gives that a shrinking gradient Ricci soliton has non-negative scalar curvature. So we deduce from equation (2.5) that

$$n(n-1)(c+H^2) \geq |\Phi|^2. \tag{3.4}$$

Whence, we obtain $c+H^2 \geq 0$ occurring equality if and only if $AX = HX$ for all $X \in \mathfrak{X}(M)$. Therefore, if this equality occurs we use (2.3) to conclude that M^n is an Einstein manifold. This enables us to use Theorem 3 in [2] to conclude that $(M^n, g, \nabla f, \lambda)$ is the Gaussian soliton.

Next we consider the case $c+H^2 > 0$. Using again equation (3.4) we deduce the next inequality $\sup_M |\Phi| < \infty$. Now we recall the following inequality obtained in [1]

$$\frac{1}{2}\Delta|\Phi|^2 \quad \geq \quad -|\Phi|^2(P_H(|\Phi|)), \tag{3.5}$$

where

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|x - n(c+H^2).$$

Since $H^2 + c > 0$ we have that $P_H(x)$ has a unique positive root given by

$$B_H = \frac{\sqrt{n}}{2\sqrt{n-1}}(\sqrt{n^2H^2 + 4(n-1)c} - (n-2)|H|).$$

Recently, in [6], it was proved that on every gradient Ricci soliton the full Omori-Yau maximum principle holds for the Laplacian. Therefore, since $\sup_M |\Phi| < \infty$, we may apply Omori-Yau principle to $|\Phi|$. Thus, we deduce the existence of a sequence $\{p_k\}_{k\in\mathbb{N}}$ in M^n such that

$$\lim_{k\to\infty} |\Phi|(p_k) = \sup_M |\Phi|, \quad \nabla|\Phi|(p_k) < \frac{1}{k} \quad \text{and} \quad \Delta|\Phi|(p_k) < \frac{1}{k}.$$

On the other hand, we have

$$\begin{aligned}\frac{1}{2}\Delta|\Phi|^2(p_k) &= |\Phi|(p_k)\Delta|\Phi|(p_k) + |\nabla|\Phi|(p_k)|^2 \\ &< |\Phi|(p_k)\frac{1}{k} + \frac{1}{k}.\end{aligned}$$

Using this in (3.5) we obtain

$$|\Phi|(p_k)\frac{1}{k} + \frac{1}{k} > \frac{1}{2}\Delta|\Phi|^2(p_k) \geq -|\Phi|^2(p_k)P_H(|\Phi|(p_k)).$$

Taking limits we infer

$$0 \geq -(\sup|\Phi|)^2 P_H(\sup|\Phi|),$$

that is

$$(\sup|\Phi|)^2 P_H(\sup|\Phi|) \geq 0,$$

which implies that either $\sup|\Phi| = 0$, from which we have $AX = HX$ for any $X \in \mathfrak{X}(M)$, or $\sup|\Phi| > 0$. Then we deduce $P_H(\sup|\Phi|) \geq 0$ and $\sup|\Phi| \geq B_H$. First let us consider $|\Phi| = 0$. Using (2.5) we have $R = n(n-1)(c + H^2) \geq 0$. Now we claim that $R = 0$. Indeed, since we have a Ricci soliton $Ric(\nabla f, X) = \frac{1}{2}\langle \nabla R, X\rangle$, see e.g. [2]. On the other hand, equation (2.3) gives $Ric(\nabla f, X) = \frac{R}{n}\langle \nabla f, X\rangle$. Since f is non trivial and R is constant we compare the last two identities to conclude that $R = 0$ as we wish. Since $c + H^2 > 0$ we arrive at a contradiction. Therefore, we have $\sup|\Phi| > 0$ which gives $\sup|\Phi| \geq B_H$ and we complete the proof of the theorem. □

Acknowledgement. *The first and third authors would like to thank the Department of Mathematics at Universidade Federal do Amazonas for hospitality during their visit in 2011 and 2012, where part of this work was carried out.*

References

[1] Alencar, H and do Carmo, M.: Hypersurfaces with constant mean curvature in Spheres, Proc. Amer. Math. Soc., 120 (1994) 1223-1229.

[2] Aquino, C., Barros, A. and Ribeiro Jr, E.: Some applications of the Hodge-de Rham decomposition to Ricci solitons. Results in Math. 60 (2011) 245-254.

[3] Baird, P. and Danielo, L.: Three-dimensional Ricci solitons which project to surfaces. J. Reine Angew. Math. 608, 2007, 65-71.

[4] Barros, A. and Ribeiro Jr, E.: Some characterizations for compact almost Ricci solitons. Proc. Amer. Math. Soc. 140, (2012) 1033-1040.

[5] Barros, A., Batista, R. and Ribeiro Jr, E.: Rigidity of gradient almost Ricci soliton. To appear in Illinois J. of Math. (2012).

[6] Bessa, G., Pigola, S. and Setti, A.: Spectral and stochastic properties of the f-Laplacian, solution of PDE's at infinity and geometric applications. Preprint, arxiv: 1107.1172 [math.DG] (2011). To appear in Revista Matematica Iberoamericana (2011).

[7] Caminha, A. Camargo, F. and Souza, P.: Complete foliations of space forms by hypersurfaces. Bull. Braz. Math. Soc. 41, (2010) 339-353.

[8] Cao, Huai-Dong.: Recent progress on Ricci soliton. Adv. Lect. Math. (ALM), 11 (2009), 1-38.

[9] Catino, G.: Generalized quasi-Einstein manifolds with harmonic weyl tensor. Preprint, arXiv: 1012.5405v1 [math.DG] (2010). To appear in Math. Z. (2011).

[10] Chen, B.: Strong uniqueness of the Ricci flow. J. Diff. Geom. 82 (2009), 363-382.

[11] Chern, S. S., do Carmo, M. and Kobayashi, S.: Minimal submanifolds of a sphere with second fundamental form of constant length, Functional Analysis and Related Fields(F. Browder, ed.), Springer-Verlag, Berlin, 1970, 59-75.

[12] Frankel, T.: On the fundamental group of a compact minimal submanifold, Ann. of Math. 83 (1966), 68-73.

[13] Kuiper, N.: On Conformally flat spaces in the large, Ann. of Math. 50 (1949), 916-924.

[14] Lawson, B.: Local rigidity theorems for minimal hypersurfaces, Ann. of Math. (2) 89 (1969), 187-197.

[15] Lott, J.: On the long time behavior of type-III Ricci flow solutions. Math. Ann. 339, (2007), 627-666.

[16] Mastrolia, P., Rigoli, M. and Rimoldi, M. Some geometric analysis on generic Ricci soliton. Preprint, arXiv: 1107.2829v1 [math.DG], (2011).

[17] Naber, A.: Noncompact shrinking 4-solitons with nonnegative curvature. Preprint, arXiv 0710.5579 [math.DG], (2007).

[18] Perelman, G. Ya.: The entropy formula for the Ricci flow and its geometric applications. arXiv math/0211159 [math.DG], (2002).

[19] Pigola, S., Rigoli, M., Rimoldi, M. and Setti, A.: Ricci Almost Solitons. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) Vol. X (2011), 757-799.

[20] Simons, J., Minimal varieties in Riemannian manifolds, Ann. of Math. (2) 88, (1968), 62-105.

[21] Yau-S.T.: Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry. Indiana Univ. Math.J., 25 (1976), 659-670.

[22] Wylie, W.: Complete shrinking Ricci solitons have finite fundamental group. Proc. Amer. Math. Soc. 136 (2008), 1803-1806.

[1] Departamento de Matemática
Universidade Federal do Ceará
60455-760-Fortaleza-CE-Brazil
e-mail: abdenago@pq.cnpq.br

[2] Current: Departamento de Matemática
Universidade Federal do Ceará,
60455-760-Fortaleza-CE-Brazil
Permanent: Departamento de Matemática-UFAM
69077-070-Manaus-AM-Brazil
e-mail: jn_vg@hotmail.com

[3] Departamento de Matemática
Universidade Federal do Ceará
60455-760-Fortaleza-CE-Brazil
e-mail: ernani@mat.ufc.br