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Orbital stability of periodic travelling wave solutions of the modified Boussinesq equation

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Dedicated to Professor Gervásio Colares on the occasion of his 80th birthday.

Abstract

This paper is concerned with stability of periodic travelling wave solutions of the modified Boussinesq equation. It will be shown that the constants and a nontrivial class of these solutions are nonlinearly stable in the energy space for a range of their speeds of propagation and periods.

1 Introduction

The original Boussinesq equations was first derived in 1871 and are among the first models for nonlinear, dispersive wave propagation [10, 11]. These evolution equations possess special travelling wave solutions known as Scott Russel's solitary waves or *solitons* [2, 9, 19], Boussinesq's and Korteweg-de Vries *cnoidal waves* [5, 6, 17, 7, 18] and *dnoidal waves* (see Section 3 below). The cnoidal and dnoidal wave solutions are periodic travelling waves written in terms of the Jacobi elliptic functions.

We consider the modified Boussinesq partial differential equation

$$u_{tt} - u_{xx} + (u^3 + u_{xx})_{xx} = 0. \tag{1.1}$$

The latter equation has the following equivalent form as a Hamiltonian system

$$\begin{cases} u_t = v_x \\ v_t = (u - u_{xx} - u^3)_x \end{cases} \tag{1.2}$$

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for $x \in \mathbb{R}$, $t > 0$. Here subscripts t and x connote partial differentiation with respect to t and x.

Equation (1.1) conserves energy, namely the integral

$$H(u,v) = \frac{1}{2}\int_0^L (u^2 + v^2 + u_x^2 - \frac{u^4}{2})dx, \tag{1.3}$$

does not depend on the time t. Another conserved quantity is the momentum

$$I(u,v) = \int_0^L uvdx \tag{1.4}$$

which turns out to be a relevant quantity in the investigation of stability properties of travelling waves.

To make precise the notion of stability we use, let τ_s be the translation by s, $\tau_s\phi(x) = \phi(x+s)$ for $x \in \mathbb{R}$ and let $\overrightarrow{\phi} = (\phi_c(x-ct), \psi_c(x-ct))$ be an L-periodic travelling wave solution to system (1.2), where $\phi_c : \mathbb{R} \to \mathbb{R}$, $\psi_c : \mathbb{R} \to \mathbb{R}$, $L > 0$ is the period of ϕ_c and ψ_c and c is the wave's speed of propagation. If we define the $\overrightarrow{\phi}$-orbit to be the set $\Omega_{\overrightarrow{\phi}} = \{\overrightarrow{\phi}(\cdot + s), \;\; s \in \mathbb{R}\}$, $\overrightarrow{\phi}$ is called orbitally stable if profiles near its orbit remain near the orbit for as long as it exists.

So, we have the following definition. Let X be a Hilbert space.

Definition 1. (Orbital Stability) *Let* $\overrightarrow{\phi} = (\phi_c(x-ct), \psi_c(x-ct)) \in X$ *be an L-periodic travelling wave solution to system (1.2). We say that the orbit* $\Omega_{\overrightarrow{\phi}}$ *is stable in the X sense by the flow of system (1.2) if for each* $\epsilon > 0$ *there exists* $\delta = \delta(\epsilon) > 0$ *such that if* $\overrightarrow{u}_0 \in X$ *and* $\inf_{s\in\mathbb{R}} ||\overrightarrow{u}_0 - \tau_s(\overrightarrow{\phi})||_X < \delta$ *then the solution* $\overrightarrow{u}(t)$ *of (1.2) with* $\overrightarrow{u}(0) = \overrightarrow{u}_0$ *satisfies, for all t for which* $\overrightarrow{u} = (u,v)$ *exists,*

$$\inf_{s\in\mathbb{R}} ||\overrightarrow{u}(t) - \tau_s(\overrightarrow{\phi})||_X < \epsilon. \tag{1.5}$$

Otherwise, we say that $\Omega_{\overrightarrow{\phi}}$ is X-unstable.

Here, $X := H^1_{per}([0,L]) \times L^2_{per}([0,L])$. (The choice of norm in (1.5) is dictated by the form of the Hessian or "linearized Hamiltonian" $H''(\overrightarrow{\phi}) + cI''(\overrightarrow{\phi})$ and varies from problem to problem.)

Inserting the L-periodic travelling wave solution $\overrightarrow{\phi}_c = (\phi_c(x-ct), \psi_c(x-ct))$ in (1.2) leads to the system

$$\begin{cases} -c\phi_c'(\xi) = \psi_c'(\xi) \\ -c\psi_c'(\xi) = (\phi_c - \phi_c'' - \phi_c^3)'(\xi), \end{cases}$$

where $'$ connotes $\frac{d}{d\xi}$ and $\xi = x - ct$. Integrating the latter system, we obtain the nonlinear system

$$\begin{cases} -c\phi(\xi) = \psi(\xi) + K_1 \\ -c\psi(\xi) = \phi(\xi) - \phi''(\xi) - \phi^3(\xi) + K_2, \end{cases}$$

where K_1, K_2 are integration constants, which will be considered equal to zero here. Then, we obtain

$$(H' + cI')(\overrightarrow{\phi}_c) = 0. \tag{1.6}$$

Next observe that relation (1.6) characterizes $\overrightarrow{\phi}_c = (\phi_c, \psi_c)$ as a critical point of H subject to the constraint $I(u, v) = I(\phi_c, \psi_c)$. In order to prove stability for $\overrightarrow{\phi}_c$ we will examine the relation between the convexity properties of the function

$$d(c) = H(\overrightarrow{\phi}_c(\cdot)) + cI(\overrightarrow{\phi}_c(\cdot)), \tag{1.7}$$

and the properties of the functional H near the critical point $\overrightarrow{\phi}_c$ under the constraint $I = constant$.

Bona and Sachs in [9] proved that the well known solitary waves $\overrightarrow{\phi} = (\phi_c(x - ct), \psi_c(x - ct))$ of the generalized Boussinesq equation

$$\begin{cases} u_t = v_x \\ v_t = (u - u_{xx} - u^p)_x \end{cases} \tag{1.8}$$

are stable in the $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ norm for speeds c such that $\frac{p-1}{4} < c^2 < 1$. Differently from the solitary wave solutions case, we do not know explicit periodic travelling wave solutions in the x-variable for the system (1.8) for every p. For this reason, we will treat here only the case $p = 3$. Regarding the classical case $p = 2$, in [5] the author proves nonlinear stability properties of L-periodic cnoidal wave solutions in the energy space $H^1_{per}([0, L]) \times L^2_{per}([0, L])$, by periodic disturbances with period L. In [4], Angulo & Quintero showed that special periodic travelling wave solutions of an one-dimensional Boussinesq-type equation are orbitally stable in a closed subspace $\{u \in H^1_{per}([0, L]);\ \int_0^L u dx = 0\}$ of the energy space, for a range of their speeds of propagation and periods.

In this paper, we first observe the existence of the nonzero trivial solutions $(\pm\sqrt{1 - c^2}, \mp c\sqrt{1 - c^2})$ for system (1.6). Then, we prove the following result of stability for these solutions, which follows from the sufficiency part of Theorem 2 of [13].

Theorem 1. *Let $c \in (-1,1)$ and $L > 0$. Then the constant solutions $(\pm\sqrt{1-c^2}, \mp c\sqrt{1-c^2})$ are X-stable with regard to the flow of the modified Boussinesq equation, provided $c^2 > \frac{1}{3}$ and $1-c^2 < \frac{2\pi^2}{L^2}$.*

Next, we show the existence of a smooth curve $c \mapsto \overrightarrow{\phi}_c = (\phi_c, \psi_c)$ of dnoidal wave solutions to system (1.2), with a fixed period L (Theorem 4 below). Then, orbital stability of these solutions is established in X for a certain range of their speeds of propagation and periods, as a consequence of the general stability criterion given by M. Grillakis, J. Shatah and W. Strauss [13]. More precisely, our main result regarding stability of the dnoidal waves $\overrightarrow{\phi}_c$, given by Theorem 4 below, is the following:

Theorem 2. (Stability Theorem) *Let $c \in (-1,1)$ and $L > \sqrt{5}\pi$. Then there exists a $\frac{2\pi^2}{L^2} < c_0 \leq \frac{2}{5}$, such that the orbit $\Omega_{\overrightarrow{\phi}_c}$ is X-stable with respect to the flow of the modified Boussinesq equation, provided $c_0 > 1-c^2 > \frac{2\pi^2}{L^2}$.*

Remark 1.1. *Some related results with Theorem 2 can be found in [21].*

The outline of the proof is as follows. First, we prove local existence of smooth solutions for the initial value problem (1.1) with initial conditions

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x). \tag{1.9}$$

Then, the nonlinear stability of the special periodic travelling wave solutions of this equation, given by Theorem 4, is proved by using the criterion given by Grillakis *et al.* [13]. This criterion is based on the convexity property of the classical function $d(c)$ given by (1.7), and on a spectral analysis of the operator linearized around $\overrightarrow{\phi}$, which we denote by $\mathcal{L}_c$. Specifically, by using Floquet's theory, we show that the first three eigenvalues β_0, β_1 and β_2 of $\mathcal{L}_c$ are simple, $\beta_1 = 0$ and the corresponding eigenfunction is $\frac{d}{dx}\overrightarrow{\phi}$; moreover, the rest of the spectrum is bounded away from zero (see Section 4). We remark that stability of $\overrightarrow{\phi}$ is established with respect to perturbations of periodic functions of the same period L in X. Finally, local existence coupled with the stability result is shown to imply the conditions that lead to global existence, at least for initial data close to the stable dnoidal wave.

2 Stability of the constant solutions

Unlike the situation that arises for solitary waves where the natural physically relevant assumption is that $\phi_c(z) \to 0$ as $z \to \pm\infty$, for which the only trivial solution is $\phi_c \equiv 0$, the dnoidal-wave problem throws up two trivial solutions. This is easily appreciated from (1.6) (or (3.11)). In fact, equation (3.11) possess two nonzero constant solutions given by $\phi_0 = \pm\sqrt{1-c^2}$ and so $(\pm\sqrt{1-c^2}, \mp c\sqrt{1-c^2})$ are solutions to system (1.6).

Proof. (of Theorem 1.1) Consider the periodic eigenvalue problem on [0,L]

$$\begin{cases} \mathcal{L}_0(f,g) = \lambda(f,g), \\ f(0) = f(L), \quad f'(0) = f'(L), \\ g(0) = g(L), \quad g'(0) = g'(L), \end{cases} \tag{2.1}$$

where $\mathcal{L}_0 := \begin{pmatrix} 1-\partial_{xx}-3\phi_0^2 & c \\ c & 1 \end{pmatrix}$, $(f,g) \in D(\mathcal{L}_0)$ and show for $c \neq 0$ that

$$\lambda_n^{\pm} = \frac{2-3(1-c^2)+\left(\frac{2n\pi}{L}\right)^2 \pm \sqrt{\left[3(1-c^2)-\left(\frac{2n\pi}{L}\right)^2\right]^2 - 4(1-c^2)+4}}{2},$$

$n \geq 0$ are the corresponding eigenvalues. We observe that the first eigenvalue $\lambda_0^- < 0$ for all $c \in (-1,1)$, $c \neq 0$ and $\lambda_1^- > 0 \Leftrightarrow 1-c^2 < \frac{2\pi^2}{L^2}$. Actually, with this last restriction $(\lambda_n^{\pm}) > 0 \ \ \forall n \geq 1$. In the case where $c = 0$, problem (2.1) becomes an easier problem, from which we deduce that $\lambda_n = \left(\frac{2n\pi}{L}\right)^2 - 2$ are the eigenvalues. Then, in this situation $\lambda_0 = 2$ and $\lambda_1 > 0 \Leftrightarrow 1-c^2 < \frac{2\pi^2}{L^2}$.

On the other hand,

$$\begin{aligned} d''(c) =& \frac{d}{dc} I(\phi_0, -c\phi_0) = -\frac{d}{dc}\left(c\int_0^L \phi_0^2 dx\right) \\ =& (-1+3c^2)L > 0 \Leftrightarrow c^2 > 1/3. \end{aligned}$$

□

Remark 2.1. *Note that we have a global existence result in X for solutions of system (1.2) with initial data close to* $(\phi_0, -c\phi_0)$*, similar to Theorem 6 below.*

3 Stability of the nontrivial solutions

3.1 Local Existence Theory

In the present section a theorem asserting the local well-posedness of the initial value problem (1.1)-(1.9) is stated. The well-posedness theorem is a straightforward consequence of the abstract techniques of Kato [15, 16] for quasi-linear evolution equations, and consequently the proof is omitted.

To apply Kato's theory to the initial value problem (1.1)-(1.9), we consider the equivalent formulation (1.2)-(3.1) with

$$\begin{cases} u(x,0) = u_0(x) \\ v(x,0) = v_0(x) \end{cases} \tag{3.1}$$

for $x \in \mathbb{R}$.

For $T > 0$ and $s \in \mathbb{R}$ define the following spaces of solutions and initial conditions

$$\begin{cases} X_s(T) = C(0,T; H^{s+2}_{per}([0,L])) \cap C^1(0,T; H^s_{per}([0,L])) \\ Y = H^{s+2}_{per}([0,L]) \times H^{s+1}_{per}([0,L]). \end{cases} \tag{3.2}$$

Theorem 3. *Let $(u_0, v_0) \in Y$ for some $s > 1/2$. Then there exists $T > 0$ which depends only upon $||(u_0, v_0)||_Y$, and unique functions $u \in X_s(T)$ and $v \in X_{s-1}(T)$, which solve the initial value problem* $(1.2)-(3.1)$. *Moreover, the pair (u, v) depends continuously on (u_0, v_0) in the sense that the associated mapping $(u_0, v_0) \to (u, v)$ is continuous from Y into the space $X_s(T) \times X_{s-1}(T)$.*

This theorem follows directly from the general results of Kato (1974, 1983) on quasi-linear evolution equations. The functional-analytic setting for Kato's theory consists of a pair of reflexive Banach spaces X and Y, with Y continuously and densely imbedded in X. A central role in the theory is played by a Banach-space isomorphism S of Y onto X, and the norms on these two spaces are chosen in such a way that S is an isometry. The theory applies to the abstract, quasi-linear evolution equation

$$\overrightarrow{U}_t + A(t, \overrightarrow{U})\overrightarrow{U} = F(t, \overrightarrow{U}), \quad \text{for} \quad t > 0 \quad \text{with} \quad \overrightarrow{U}(0) = \overrightarrow{\phi}, \tag{3.3}$$

where $\overrightarrow{\phi}$ is a given initial value. The theory asserts that there exists a positive time T such that (3.3) possesses a unique solution in $C(0,T;Y) \cap C^1(0,T;X)$, under certain assumptions.

To apply Kato's theory to the situation envisaged in Theorem 3, take $X = H^s_{per}([0,L]) \times H^{s-1}_{per}([0,L])$ with $s > 1/2$, and take Y as in (3.2). Also, let $S = (I_d - \partial_x^2, I_d - \partial_x^2)$ with I_d denoting the identity operator, let A be the matrix of differential operators

$$A = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x + \partial_x^3 & 0 \end{pmatrix}, \tag{3.4}$$

and take the nonlinear operator F to be

$$F = F(t, u, v) = \begin{pmatrix} 0 \\ -(u^3)_x \end{pmatrix}. \tag{3.5}$$

With this choice of A and F, and writing

$$\overrightarrow{U} = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.6}$$

(3.3) reduces to (1.2) – (3.1) if $\overrightarrow{\phi} = (u_0, v_0)$, and it is straightforward to verify that the hypotheses required in Kato's theory are satisfied.

A consequence of Theorem 3 is stated in the following corollary. Define for $T > 0$ and $s \in \mathbb{R}$

$$Y_s(T) = X_s(T) \cap C^2(0, T, H_{per}^{s-2}([0, L])). \tag{3.7}$$

Corollary 1. *Let $(u_0, v_0) \in Y$ for some $s > 1/2$. Then there exists $T > 0$ which depends only upon $||(u_0, v_0)||_Y$, and a unique function $u \in Y_s(T)$ which is a solution of Eq. (1.1) in the distributional sense on $\mathbb{R} \times [0, T]$, and for which $u(\cdot, 0) = u_0$ and $u_t(\cdot, 0) = v_0'$. The solution u depends continuously on (u_0, v_0) in the sense that the associated mapping $(u_0, v_0) \to u$ is continuous from Y into the space $Y_s(T)$.*

Remark 3.1. *If $s > 5/2$ then the solution is classical, which means that all derivatives featured in the equation exist pointwise and are jointly continuous functions of x and t.*

3.2 Existence of a smooth curve of dnoidal wave solutions with a fixed period L for the system (1.2)

This section is devoted to establish the existence of a smooth curve of periodic travelling wave solutions for the system (1.2), which are solutions of the form

$$\overrightarrow{u}(x, t) = (u(x, t), v(x, t)) = (\phi(x - ct), \psi(x - ct)). \tag{3.8}$$

Substituting (3.8) in (1.2) leads to the system

$$\begin{cases} -c\phi'(\xi) = \psi'(\xi) \\ -c\psi'(\xi) = (\phi - \phi'' - \phi^3)'(\xi), \end{cases} \tag{3.9}$$

where $'$ denotes $\frac{d}{d\xi}$ and $\xi = x - ct$. Integrating (3.9), we obtain the nonlinear system

$$\begin{cases} -c\phi(\xi) = \psi(\xi) + K_1 \\ -c\psi(\xi) = \phi(\xi) - \phi''(\xi) - \phi^3(\xi) + K_2, \end{cases} \tag{3.10}$$

where K_1, K_2 are integration constants, which will be considered equal to zero here. Then, ϕ must satisfy

$$\phi'' - w\phi + \phi^3 = 0, \tag{3.11}$$

where $w = w(c) = 1 - c^2$ will be considered positive.

Next, we show how to construct a smooth curve of solutions for (3.11) with a fixed fundamental period L, and depending on the parameter c. In order to do this we first observe from (3.11) that ϕ satisfies the first order equation

$$\begin{aligned}(\phi')^2 &= \frac{1}{2}\big[-\phi^4 + 2w\phi^2 + 4B_\phi\big] \\ &= \frac{1}{2}(\eta_1^2 - \phi^2)(\phi^2 - \eta_2^2),\end{aligned} \tag{3.12}$$

where B_ϕ is an integration constant and $-\eta_1$, η_1, $-\eta_2$, η_2 are the real zeros of the polynomial $p_\phi(t) = -t^4 + 2wt^2 + 4B_\phi$, which satisfy the relations

$$\begin{cases} 2w &= \eta_1^2 + \eta_2^2 \\ 4B_\phi &= -\eta_1^2\eta_2^2. \end{cases} \tag{3.13}$$

Moreover, we assume without lost of generality that $\eta_1 > \eta_2 > 0$ and we obtain from (3.12) that $\eta_2 \le \phi \le \eta_1$. By defining $\varphi = \frac{\phi}{\eta_1}$ and $k^2 = \frac{(\eta_1^2 - \eta_2^2)}{\eta_1^2}$, (3.12) becomes $(\varphi')^2 = \frac{\eta_1^2}{2}(1-\varphi^2)(\varphi^2 - 1 + k^2)$. We also impose the crest of the wave to be at $\xi = 0$, that is $\phi(0) = 1$. Now, we define a further variable ψ via the relation $\varphi^2 = 1 - k^2\sin^2\psi$ and so we get that $(\psi')^2 = \frac{\eta_1^2}{2}(1 - k^2\sin^2\psi)$. Then we obtain for $l = \frac{\eta_1}{\sqrt{2}}$ that $\int_0^{\psi(\xi)} \frac{dt}{\sqrt{1-k^2\sin^2 t}} = l\xi$. Therefore, from the definition of the Jacobian elliptic function $y = \mathrm{sn}(u; k)$ (see in the Appendix or in Byrd & Friedman [8]) we can write the last equality as $\sin\psi = \mathrm{sn}(l\xi; k)$ and hence $\varphi(\xi) = \sqrt{1 - k^2\mathrm{sn}^2(l\xi; k)} = \mathrm{dn}(l\xi; k)$. Returning to the initial variable, we obtain the called *dnoidal wave solution* associated to the equation (3.11),

$$\phi(\xi) \equiv \phi(\xi; \eta_1, \eta_2) = \eta_1 \mathrm{dn}\Big(\frac{\eta_1}{\sqrt{2}}\xi; k\Big) \tag{3.14}$$

with

$$k^2 = \frac{\eta_1^2 - \eta_2^2}{\eta_1^2}, \qquad \eta_1^2 + \eta_2^2 = 2w, \qquad 0 < \eta_2 < \eta_1. \tag{3.15}$$

Next, dn has fundamental period $2K$, $\mathrm{dn}(u + 2K; k) = \mathrm{dn}(u; k)$, where $K = K(k)$ represents the complete elliptic integral of the first kind (see Appendix);

it follows that the dnoidal wave ϕ in (3.14) has fundamental period, T_ϕ, given by

$$T_\phi \equiv \frac{2\sqrt{2}}{\eta_1} K(k). \tag{3.16}$$

Now, we show that $T_\phi > \frac{\sqrt{2}\pi}{\sqrt{w}}$. First, we express T_ϕ as a function of η_2 and w. In fact, for every $\eta_2 \in (0, \sqrt{w})$ there is a unique $\eta_1 \in (\sqrt{w}, \sqrt{2w})$ satisfying the first relation in (3.13), namely, $\eta_1 = \sqrt{2w - \eta_2^2}$. So, from (3.16) we obtain

$$T_\phi(\eta_2, w) = \frac{2\sqrt{2}}{\sqrt{2w - \eta_2^2}} K(k(\eta_2)) \quad \text{with} \quad k^2(\eta_2, w) = \frac{2w - 2\eta_2^2}{2w - \eta_2^2}. \tag{3.17}$$

Then, by fixing $w > 0$, we have that $T_\phi \to +\infty$ as $\eta_2 \to 0$ and $T_\phi(\eta_2) \to \frac{\pi\sqrt{2}}{\sqrt{w}}$ as $\eta_2 \to \sqrt{w}$. So, since the mapping $\eta_2 \mapsto T_{\phi_w}(\eta_2)$ is strictly decreasing (see proof of Proposition 1), it follows that $T_\phi > \frac{\sqrt{2}\pi}{\sqrt{w}}$.

Now, we obtain a dnoidal wave solution with period L. For $w_0 > \frac{2\pi^2}{L^2}$ there is a unique $\eta_{2,0} \in (0, \sqrt{w_0})$ such that $T_\phi(\eta_{2,0}, w_0) = L$. So, for $\eta_{1,0}$ such that $\eta_{1,0}^2 + \eta_{2,0}^2 = 2w_0$, the dnoidal wave $\phi(\cdot) = \phi(\cdot, \eta_{1,0}, \eta_{2,0})$ has fundamental period L and satisfies (3.11) with $w = w_0$.

By the above analysis the dnoidal wave $\phi(\cdot, \eta_1, \eta_2)$ in (3.14) is completely determined by w and η_2 and will be denoted by $\phi_w(\cdot; \eta_2)$ or ϕ_w.

The next result, which corresponds to Theorem 2.1 and Corollary 2.2 in [3], is concerned with the existence of a smooth curve of dnoidal wave solutions for equation (3.11).

Proposition 1. *Let $L > 0$ be arbitrary but fixed. Consider $w_0 > \frac{2\pi^2}{L^2}$ and the unique $\eta_{2,0} = \eta_2(w_0) \in (0, \sqrt{w}_0)$ such that $T_{\phi_{w_0}} = L$. Then,*

(1) *there exist an interval $\mathcal{I}(w_0)$ around w_0, an interval $J(\eta_{2,0})$ around $\eta_{2,0}$ and a unique smooth function $\Lambda : \mathcal{I}(w_0) \to J(\eta_{2,0})$ such that $\Lambda(w_0) = \eta_{2,0}$ and*

$$\frac{2\sqrt{2}}{\sqrt{2w - \eta_2^2}} K(k) = L, \tag{3.18}$$

where $w \in \mathcal{I}(w_0)$, $\eta_2 = \Lambda(w)$ and $k^2 = k^2(w) \in (0, 1)$ is defined by (3.17).

(2) *The dnoidal wave solution in (3.14), $\phi_w(\cdot; \eta_1, \eta_2)$, determined by $\eta_1 \equiv \eta_1(w) = \sqrt{2w - \eta_2^2}$, $\eta_2 \equiv \eta_2(w)$, has fundamental period L and satisfies (3.11). Moreover, the mapping $w \in \mathcal{I}(w_0) \to \phi_w \in H^1_{per}([0, L])$is a smooth function.*

(3) *$\mathcal{I}(w_0)$ can be chosen as* $(\frac{2\pi^2}{L^2}, +\infty)$.

(4) *The mapping* $\Lambda : (\frac{2\pi^2}{L^2}, +\infty) \to J(\eta_{2,0})$ *is strictly decreasing.*

Proof. See [3].

□

From this result we conclude the following existence theorem.

Theorem 4. *Let $L > \pi\sqrt{2}$. Then there exists a smooth curve of dnoidal wave solutions for the system (1.2) in $H^n_{per}([0,L]) \times H^m_{per}([0,L])$, $n, m \geq 0$ which satisfy the system (3.10) with integration constants $K_1 = K_2 = 0$; this curve is given, for $w(c) = 1 - c^2$, by*

$$c \in \Big(-\sqrt{1 - \frac{2\pi^2}{L^2}}, \sqrt{1 - \frac{2\pi^2}{L^2}} \Big) \to (\phi_{w(c)}, \psi_{w(c)}). \tag{3.19}$$

Moreover, $\phi_{w(c)}(\xi) = \sqrt{2w - \eta_2^2}\,\mathrm{dn}\left[\frac{\sqrt{2w-\eta_2^2}}{\sqrt{2}}\xi; k\right]$, $\psi_{w(c)} = -c\phi_{w(c)}$, *where the smooth function $\eta_2 \equiv \eta_2(w(c))$ is given by Proposition 1 and $k = k(w(c))$ by (3.17).*

3.3 Spectral Analysis

The following result is concerned with the spectral properties associated to the linear operator

$$\mathcal{L}_c = H''(\phi_{w(c)}, \psi_{w(c)}) + cI''(\phi_{w(c)}, \psi_{w(c)}) \tag{3.20}$$

determined by the periodic solutions $(\phi_{w(c)}, \psi_{w(c)})$ found in Theorem 4.

Theorem 5. *Let $\mathcal{L}_c$ be the linear operator defined on $H^2_{per}([0,L]) \times H^1_{per}([0,L])$ by (3.20). Then the first two eigenvalues β_0 and β_1 of $\mathcal{L}_c$ are simple, and satisfy $\beta_0 < \beta_1 = 0$; and $\overrightarrow{\phi}'_{w(c)}$ is the eigenfunction of β_1. Moreover, the rest of the spectrum consists of a discrete set of eigenvalues.*

Proof. See Theorem 4.2 in [6]. □

3.4 Proof of Theorem 2

In order to prove this result, we only need to show that the function $d(c)$ defined by $d(c) = H(\overrightarrow{\phi_c}) + cI(\overrightarrow{\phi_c}) \equiv \frac{1}{2}\int_0^L (\phi_c^2 + \psi_c^2 + (\phi_c)_x^2 - \frac{\phi_c^4}{4}) + c\int_0^L \phi_c\psi_c dx$, is convex.

Remark 3.2. Condition $d''(c) > 0$ is equivalent to the condition $\frac{dI(\phi_{w(c)},\psi_{w(c)})}{dc} > 0$, since

$$(H' + cI')(\phi_{w(c)}, \psi_{w(c)}) = 0.$$

So, will prove that there is a $c_0 \in (0, \frac{2}{5}]$ such that $\frac{d}{dc}I(\phi_{w(c)}, \psi_{w(c)}) > 0$, for $1 - c_0 < c^2 < 1 - \frac{2\pi^2}{L^2}$.

To this end, notice that

$$\begin{aligned} \frac{d}{dc}I(\phi_w, \psi_w) &= \frac{d}{dc}\int_0^L \phi_w\psi_w = -\int_0^L \phi_w^2 dx - c\frac{d}{dw}\Big[\int_0^L \phi_w^2 dx\Big]\frac{dw}{dc} \\ &= -\int_0^L \phi_w^2 dx + 2c^2\frac{d}{dw}\Big[\int_0^L \phi_w^2 dx\Big]. \end{aligned} \tag{3.21}$$

Now,

$$\frac{d}{dw}\frac{1}{2}\int_0^L \phi_w^2 dx = \frac{4}{L}\frac{d}{dk}\big[K(k)E(k)\big]\frac{dk}{dw} > 0. \tag{3.22}$$

Indeed, we observe from (3.14), (3.15) and (3.18) that

$$||\phi_w||^2 = \sqrt{2}\eta_1\int_0^{\frac{\eta_1}{\sqrt{2}}L} dn^2(x;k)dx = \frac{8K(k)}{L}\int_0^K dn^2(x;k)dx, \tag{3.23}$$

where we used that the Jacobi elliptic function dn has fundamental period $2K$ and is an even function. Now, by using that $\int_0^K cn^2(x;k)dx = \frac{1}{k^2}\big[E(k) - (k')^2K(k)\big]$ and $dn^2(x;k) = 1 - k^2 + k^2cn^2(x;k)$, it follows from (3.23) that

$$\frac{1}{2}\int_0^L \phi_w^2 dx = \frac{4}{L}K(k)E(k). \tag{3.24}$$

Now, Proposition 1 and Theorem 1 implies that the map $w \to \Lambda(w) \equiv \eta_2(w)$ is strictly decreasing and from (3.17), with $\eta_2 = \eta_2(w)$, we have that

$$\frac{dk}{dw} = \frac{1}{2k}\Big[\frac{2\eta_2^2 - 4w\eta_2\eta_2'}{(2w - \eta_2^2)^2}\Big] > 0. \tag{3.25}$$

Thus, since $k \in (0,1) \to K(k)E(k)$ is strictly decreasing (see Appendix), the claim (3.22) follows from (3.24) and (3.25).

So, from (3.21), (3.22) and (3.24), we get

$$\frac{d}{dc}I(\phi_w, \psi_w) = -\frac{8}{L}K(k)E(k) + \frac{16c^2}{L}\frac{d}{dk}\big[K(k)E(k)\big]\frac{dk}{dw}. \tag{3.26}$$

Now, considering the function Ψ defined by (2.12) in [3] and using (3.25), we obtain

$$\begin{aligned}\frac{\partial\Psi}{\partial w} &= \frac{2\sqrt{2}\sqrt{2w-\eta_2^2}\frac{dK}{dw}(k(\eta_2,w)) - 2\sqrt{2}K(k(\eta_2,w))(2w-\eta_2^2)^{-\frac{1}{2}}}{(2w-\eta_2^2)} \\ &= \frac{2\sqrt{2}\big[\sqrt{2w-\eta_2^2}\frac{dK}{dk}(k(\eta_2,w))\frac{\eta_2^2}{k(2w-\eta_2^2)^2} - K(k(\eta_2,w))(2w-\eta_2^2)^{-\frac{1}{2}}\big]}{(2w-\eta_2^2)},\end{aligned}$$

hence

$$\frac{dk}{dw} = \frac{1}{2k(2w-\eta_2^2)^2}\Big\{2\eta_2^2 - 4w\Big[\frac{k(2w-\eta_2^2)K - \eta_2^2\frac{dK}{dk}}{k(2w-\eta_2^2)K - 2w\frac{dK}{dk}}\Big]\Big\} > 0. \tag{3.27}$$

From (3.26), (3.27) and using that $2w-\eta_2^2 = \frac{\eta_2^2}{(k')^2}$, we obtain

$$\begin{aligned}\theta\frac{L}{8}\frac{dI(\phi_w,\psi_w)}{dc} &= \theta\big\{-EK + 2c^2\frac{E^2-k'^2K^2}{kk'^2}\frac{dk}{dw}\big\} \\ &= -\theta EK + 2c^2(E^2-k'^2K^2)K(\eta_2^2-2w)(k')^2 \\ &= K\big\{-\eta_2^2E(k^2\eta_2^2K - 2wE + 2wk'^2K)\big\} \\ &+ K\big\{2c^2(E^2-k'^2K^2)(k')^2(\eta_2^2-2w)\big\},\end{aligned}$$

or equivalently,

$$\begin{aligned}\theta\frac{L}{8K}\frac{dI(\phi_w,\psi_w)}{dc} &= -\eta_2^2E(k^2\eta_2^2K - 2wE + 2wk'^2K) \\ &+ 2c^2(k')^2(E^2-k'^2K^2)(\eta_2^2-2w) \\ &= \eta_2^2(-2wk'^2 - \eta_2^2k^2)EK \\ &+ (2c^2(k')^2\eta_2^2 - 4c^2w(k')^2 + 2w\eta_2^2)E^2 \\ &+ 2c^2(k')^2\eta_2^2K^2,\end{aligned} \tag{3.28}$$

where $\theta = \eta_2^2(k^2\eta_2^2K - 2wE + 2wk'^2K) < 0$.

Now, given that $(k')^2 = \frac{\eta_2^2}{2w-\eta_2^2}$, we rewrite the coefficient of E^2 in (3.28) as

$$2c^2(k')^2\eta_2^2 - 4c^2w(k')^2 + 2w\eta_2^2 = 2c^2\frac{\eta_2^4}{2w-\eta_2^2} - 4c^2w\frac{\eta_2^2}{2w-\eta_2^2} + 2w\eta_2^2$$

$$= \frac{2c^2\eta_2^4 - 4c^2w\eta_2^2 + 4w^2\eta_2^2 - 2w\eta_2^4}{2w-\eta_2^2} = \frac{\eta_2^2(2w-\eta_2^2)(2w-2c^2)}{2w-\eta_2^2} = \eta_2^2(2w-2c^2).$$

Also, the coefficient of EK can be rewritten as $\eta_2^2(-2wk'^2 - \eta_2^2 k^2) = 2\eta_2^4$. Thus,

$$\frac{L}{8K}\frac{dI(\phi_w,\psi_w)}{dc} = \frac{-2\eta_2^4 EK + 2\eta_2^2(w-c^2)E^2 + 2c^2(k')^2\eta_2^2 K^2}{\eta_2^2(k^2\eta_2^2 K - 2wE + 2wk'^2 K)}. \tag{3.29}$$

We remark that we can write w as a function of complete elliptic integrals. In fact, by integrating the equation (3.11) from 0 to L we obtain

$$w = w(c) = \frac{\int_0^L \phi_w^3(\xi)d\xi}{\int_0^L \phi_w(\xi)d\xi}, \tag{3.30}$$

which is well defined, since the solution ϕ_w is positive.

Now, using (3.14), the expression 314.01 in [8] and the fact that $F(\frac{\pi}{2};k) = K(k)$ (see Appendix), we obtain

$$\begin{aligned}\int_0^L \phi_w(\xi)d\xi &= \int_0^L \eta_1 \mathrm{dn}(\frac{\eta_1}{\sqrt{2}}\xi;k)d\xi = \sqrt{2}\int_0^{\frac{\eta_1 L}{\sqrt{2}}} \mathrm{dn}(y;k)dy \\ &= \sqrt{2}\int_0^{2K} \mathrm{dn}(y;k)dy \\ &= 2\sqrt{2}\int_0^K \mathrm{dn}(y;k)dy = \pi\sqrt{2}.\end{aligned} \tag{3.31}$$

Similarly using (3.14), the expression 314.03 in [8], and the special values $\mathrm{sn}0 = 0$, $\mathrm{sn}K = 1$, $\mathrm{cn}K = 0$ (see Appendix), it follows that

$$\begin{aligned}\int_0^L \phi_w^3(\xi)d\xi &= \int_0^L \eta_1^3 \mathrm{dn}^3(\frac{\eta_1}{\sqrt{2}}\xi;k)d\xi = \sqrt{2}\eta_1^2\int_0^{2K} \mathrm{dn}^3(y;k)d\xi \\ &= 2\sqrt{2}\eta_1^2\int_0^K \mathrm{dn}^3(y;k)d\xi \\ &= 16\sqrt{2}\frac{K^2}{L^2}\frac{1}{2}[(1+(k')^2)\frac{\pi}{2} + k^2\mathrm{sn}K\mathrm{cn}K] \\ &= 4\pi\sqrt{2}(1+(k')^2)\frac{K^2}{L^2}.\end{aligned} \tag{3.32}$$

Substituting (3.31) and (3.32) in (3.30), we deduce that

$$w(c) = 1 - c^2 = 4(1+(k')^2)\frac{K^2}{L^2}. \tag{3.33}$$

Using (3.33) and $\eta_2^2 = \frac{2w(k')^2}{1+(k')^2}$, the numerator of (3.29) will be negative if and only if $c^2(k')^2K^2 < (c^2 - w)E^2 + \eta_2^2 EK \Leftrightarrow (1-w)(k')^2K^2 < (1 -$

$2w)E^2 + \frac{2w(k')^2}{1+(k')^2}EK \Leftrightarrow w\left[2E^2 - (k')^2K^2 - \frac{2(k')^2}{1+k'^2}EK\right] < E^2 - (k')^2K^2 \Leftrightarrow$ $w < \frac{E^2-(k')^2K^2}{2E^2-(k')^2K^2-\frac{2(k')^2}{1+k'^2}EK}$.

Remark 3.3. $2E^2 - (k')^2K^2 - \frac{2(k')^2}{1+k'^2}EK > 0$ since the functions EK and $E+K$ are strictly increasing (see Appendix).

Claim:

$$\lim_{k\to 0} \frac{E^2 - (k')^2K^2}{2E^2 - (k')^2K^2 - \frac{2(k')^2}{1+k'^2}EK} = \frac{2}{5}. \tag{3.34}$$

Verification of (3.34) Indeed, denoting by $f(k) := E^2 - (k')^2K^2$ and by $g(k) := E^2 - \frac{2(k')^2}{1+k'^2}EK$, we use L'Hospital's rule to find the limit (3.34). Specifically, we show that

$$\lim_{k\to 0} f^{(j)}(k) = \lim_{k\to 0} g^{(j)}(k) = 0 \quad (j = 0,1,2,3),$$
$$\lim_{k\to 0} f^{(4)}(k) = 3\pi^2/4 \quad \text{and} \quad \lim_{k\to 0} g^{(4)}(k) = 9\pi^2/8,$$

which implies our claim.

Note that, by $\lim_{k\to 1} k'^2K^2 = 0$, we have that

$$\lim_{k\to 1} \frac{E^2 - (k')^2K^2}{E^2 - (k')^2K^2 + E^2 - \frac{2(k')^2}{1+k'^2}EK} = \frac{1}{2}. \tag{3.35}$$

Moreover, $0 < \frac{E^2-(k')^2K^2}{E^2-(k')^2K^2+E^2-\frac{2(k')^2}{1+k'^2}EK} < \frac{1}{2}$ $\forall k \in (0,1)$, since $E^2 - \frac{2k'^2}{1+k'^2}K^2 > E^2 - k'^2K^2$ $\forall k \in (0,1)$. In addition we get $(1+k'^2)K > 2E$, since the function $m(k) := (1+k'^2)K - 2E$ has the following properties: $m(0) = 0$ and $m'(k) > 0$ $\forall k \in (0,1)$. We conclude that the function

$$\frac{f(k)}{g(k)} = \frac{E^2 - (k')^2K^2}{E^2 - (k')^2K^2 + E^2 - \frac{2(k')^2}{1+k'^2}EK} \tag{3.36}$$

is strictly positive on $[0,1]$. Now, continuity plus (3.34) and (3.35) implies that $c_0 := \min_{0\le k\le 1} \frac{f(k)}{g(k)}$ satisfies $0 < c_0 \le \frac{2}{5}$.

□

3.5 A Global Existence Theorem

In this section it is shown that if the initial data (u_0, v_0) lies close enough to the initial data $(\phi_{w(c)}, \psi_{w(c)})$ corresponding to a stable dnoidal wave, then the local solution of (1.2) – (3.1), guaranteed by Theorem 3, admits a unique extension to a global smooth solution. The precise statement is as follows.

Theorem 6. *Let $L^2 > 5\pi^2$ such that $1-c^2 > 2\pi^2/L^2$ and $c \in (-1, -1+c_0) \cup (1-c_0, 1)$. Let $(\phi_{w(c)}, \psi_{w(c)})$ denote a dnoidal wave solution of* (1.2)–(3.1)*, with $w(c) = 1 - c^2$. Then there exists $\delta = \delta(c) > 0$ such that for all $(u_0, v_0) \in Y$, and $\vartheta \in \mathbb{R}$ such that*

$$||u_0(\cdot) - \phi_{w(c)}(\cdot + \vartheta)||_1 + ||v_0(\cdot) - \psi_{w(c)}(\cdot + \vartheta)||_0 \le \delta, \tag{3.37}$$

the solution (u, v) of (1.2) – (3.1) *corresponding to the initial data (u_0, v_0) is global and lies in $X_s(T) \times X_{s-1}(T)$ for all positive T. Moreover, for all $T > 0$, the mapping sending (u_0, v_0) to the solution (u, v) of* (1.2) – (3.1) *is continuous from Y into $X_s(T) \times X_{s-1}(T)$.*

Proof. Let T^* be the maximal time of existence of the solution (u, v). The goal is to show that $T^* = +\infty$. It suffices to show that the pair (u, v) remains bounded in X for all $0 \le t \le T < T^*$ with bound independent of T. This is true for all initial values sufficiently close to a stable dnoidal wave by Theorem 2. Thus the proof is finished. □

4 Appendix

In this Appendix we recall some properties of the Jacobian elliptic integrals that have been used in this work (see [8]).

First, we define *the normal elliptic integral of the first and second kinds*, $F(\varphi, k) := \int_0^\varphi \frac{d\theta}{\sqrt{1-k^2\ \sin^2\theta}} = \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$ and $E(\varphi, k) := \int_0^\varphi \sqrt{1-k^2\ \sin^2\theta}\, d\theta = \int_0^y \sqrt{\frac{1-k^2t^2}{1-t^2}}dt$, respectively, where $y = \sin\varphi$.

In their algebraic forms, these two integrals possess the following properties: the first is finite for all real (or complex) values of y, including infinity; the second has a simple pole of order 1 for $y = \infty$. The number k is called the *modulus*. This number may take any real or imaginary value. Here we wish to take $0 < k < 1$. The number k' is called the *complementary modulus* and is related to k by $k' = \sqrt{1-k^2}$. The variable φ is the *argument* of the normal elliptic integrals. When $y = 1$, the integrals above are said to be *complete*. In this case, one writes: $F(\pi/2, k) \equiv K(k) \equiv K$, and $E(\pi/2, k) \equiv E(k) \equiv E$.

Some special values of K and E are: $K(0) = E(0) = \pi/2$, $E(1) = 1$ and $K(1) = +\infty$. For $k \in (0,1)$, one has $K'(k) > 0$, $K''(k) > 0$, $E'(k) < 0$, $E''(k) < 0$ and $E(k) < K(k)$. Moreover, $E(k) + K(k)$ and $E(k)K(k)$ are strictly increasing functions in $(0,1)$.

Now, we give some derivatives of the complete elliptic integrals K and E and some important limits involving these functions, that we used in this work (cf. [1] ou [8]):

$$\frac{dK}{dk} = \frac{E - k'^2 K}{kk'^2}; \tag{4.1}$$

$$\frac{dE}{dk} = \frac{E - K}{k}; \tag{4.2}$$

$$\frac{d^2E}{dk^2} = -\frac{1}{k}\frac{dK}{dk} = -\frac{E - k'^2 K}{k^2 k'^2}; \tag{4.3}$$

$$\lim_{k\to 0} \frac{E - k'^2 K}{k^2} = \lim_{k\to 0} \frac{K - E}{k^2} = \frac{\pi}{4}; \tag{4.4}$$

$$\lim_{k\to 0} \frac{dE}{dk} = \lim_{k\to 0} \frac{(E-K)}{k^2} k = 0; \tag{4.5}$$

$$\lim_{k\to 0} \frac{dK}{dk} = 0; \tag{4.6}$$

$$\lim_{k\to 0} \frac{d^2E}{dk^2} = -\lim_{k\to 0} \frac{1}{k}\frac{dK}{dk} = -\frac{\pi}{4}; \tag{4.7}$$

$$\lim_{k\to 0} \frac{d^2K}{dk^2} = \lim_{k\to 0} [\frac{1}{k'^2} K + \frac{(3k^2-1)}{kk'^2}\frac{dK}{dk}] = \frac{\pi}{4}; \tag{4.8}$$

$$\lim_{k\to 0} \frac{d^3E}{dk^3} = 0; \tag{4.9}$$

$$\lim_{k\to 0} \frac{d^3K}{dk^3} = 0; \tag{4.10}$$

$$\lim_{k\to 0} \frac{d^4E}{dk^4} = -\frac{9\pi}{16}; \tag{4.11}$$

$$\lim_{k\to 0} \frac{d^4K}{dk^4} = \frac{27\pi}{16}; \tag{4.12}$$

$$\lim_{k\to 0} \frac{1}{k}(\frac{dK}{dk})^2 = \lim_{k\to 0} \frac{1}{k'^4} \frac{(E - k'^2 K)}{k^2}\frac{dK}{dk} = 0. \tag{4.13}$$

Remark 4.1. To see that $\lim_{k\to 0} \frac{d^3K}{dk^3} = 0$, we write $K(k) = \frac{\pi}{2}\{ \sum_{r=0}^{\infty} \frac{(2r)!(2r)!}{2^{4r}(r!)^4} k^{2r}\}$ (*see* [1], page 110), and then we get

$$\frac{d^3K}{dk^3} = \frac{\pi}{2} k \{ \sum_{r\geq 2} \frac{(2r)!(2r)!}{2^{4r}(r!)^4} 2r(2r-1)(2r-2)k^{2r-2}\}, \tag{4.14}$$

where the series converges absolutely. Actually, denoting by $a_r := \frac{(2r)!(2r)!}{2^{4r}(r!)^4}2r(2r-1)(2r-2)k^{2r-2}$, we have $\frac{|a_{r+1}|}{|a_r|} = \frac{(2r+2)^2(2r+1)^3}{2^3(r+1)^3(2r-1)(2r-2)}k$, then we get $\lim_{r\to\infty}\frac{|a_{r+1}|}{|a_r|} < 1$. So, $\lim_{k\to 0}\frac{d^3K}{dk^3} = 0$.

To see that $\lim_{k\to 0}\frac{d^3E}{dk^3} = 0$, we write $E(k) = \frac{\pi}{2}\{1 - \sum_{r=1}^{\infty}\frac{(2r-2)!(2r)!}{2^{4r-1}(r-1)!(r!)^3}k^{2r}\}$ (see [1], page 110), from which we get

$$\frac{d^3E}{dk^3} = -\frac{\pi}{2}\{\sum_{r\geq 2}\frac{(2r-2)!(2r)!}{2^{4r-1}(r-1)!(r!)^3}2r(2r-1)(2r-2)k^{2r-3}\}. \tag{4.15}$$

Proceeding as before, we obtain the desired limit.

To see that $\lim_{k\to 0}\frac{d^4K}{dk^4} = \frac{27\pi}{16}$ and $\lim_{k\to 0}\frac{d^4E}{dk^4} = -\frac{9\pi}{16}$, differentiating again the series in (4.14) and (4.15), we get, $\frac{d^4K}{dk^4} = \frac{\pi}{2}\sum_{r=2}^{\infty} b_r 2r(2r-1)(2r-2)(2r-3)k^{2r-4}$ and $\frac{d^4E}{dk^4} = -\frac{\pi}{2}\sum_{r=2}^{\infty} c_r 2r(2r-1)(2r-2)(2r-3)k^{2r-4}$, where $b_r := \frac{(2r)!(2r)!}{2^{4r}(r!)^4}2r(2r-1)(2r-2)(2r-3)k^{2r-4}$ and $c_r := \frac{(2r-2)!(2r)!}{2^{4r-1}(r-1)!(r!)^3}2r(2r-1)(2r-2)(2r-3)k^{2r-4}$. It's easy to see that $\frac{d^4K}{dk^4} \to \frac{\pi}{2}b_2 4! = \frac{27\pi}{16}$ and $\frac{d^4E}{dk^4} \to -\frac{\pi}{2}c_2 4! = -\frac{9\pi}{16}$, as $k \to 0$.

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