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# Shadow boundaries in space forms

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*Dedicated to Professor Gervásio Colares on the occasion of his eightieth birthday.*

**Abstract**

We study shadow boundaries of submanifolds of Riemannian manifolds admitting a closed conformal vector field. As applications we give a method to find a principal direction in a compact hypersurface and a characterization of totally umbilical hypersurfaces in space forms.

## 1 Introduction

Given a Riemannian manifold $N$, an immersed submanifold $M$ and a vector field $Y$ in $N$, the notion of the shadow boundary of $M$ is a natural one: It is the set of points of $M$ such that $Y$ is tangent to $M$. This concept appears already in 1990, in an article [2] by J. Choe, under the name of *horizon* (see Definition 2.1 there) where he applied it to the study of minimal surfaces.

In [4], M. Ghomi investigated a very close concept, the *shadow*. Given a fied unit vector in $\mathbb{R}^3$, or equivalently, a parallel vector field $v$, the shadow of an orientable surface in $\mathbb{R}^3$ with respect to $v$ is the set of points of the surface such that the sign of the angle between $v$ and a given nowhere zero global normal vector field to the surface does not change. The relation between these concepts is that a shadow is an open set in the complement of a shadow boundary.

In a more general setting, we have to impose some conditions on the ambient space $N$ and on the vector field $Y$ in order to obtain some relevant geometrical information. For example, the paper [6] of the second named author contains some properties of shadow boundaries of submanifolds with

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respect to a parallel vector field in a Riemannian manifold.

In this work we investigate shadow boundaries of hypersurfaces with respect to a closed conformal vector field in the ambient, a concept which includes as a particular case that of a parallel vector field. In fact, we show that space forms are the first examples of manifolds admitting plenty of closed conformal vector fields; see Proposition 3.4. Our work is mainly inspired in the paper [3] about compact shadow boundaries of a hypersurface in Euclidean space. Here we extend some of the results in [3] to more general spaces, in particular to any space form.

In general, a shadow boundary is a closed subset of a hypersurface $M$. In the case here considered using closed conformal vector fields to construct shadow boundaries, we give first in Proposition 3.1 a condition on the shape operator of the immersion $M \subset N$ for a shadow boundary to be a submanifold of $M$. As a particular case of this proposition, we obtain for a hypersurface with nowhere zero Gauss-Kronecker curvature that every shadow boundary is a submanifold of $M$.

In Proposition 4.1 we give a characterization of a principal direction of a hypersurface $M$ in a space form via shadow boundaries. Namely, given a closed conformal vector field $Y$, the vector $Y_p \in T_pM$ dfines a principal direction of $M$ if and only if the (regular) shadow boundary $S\partial(M, Y)$ is orthogonal to the vector field $Y$. A useful corollary of this result (see Corollary 3.8) says that a surface in a three dimensional manifold with the property that every shadow boundary is a line of curvature must be totally umbilical.

Our main result here is Theorem 4.5 and relates shadow boundaries to the geometry of a submanifold: Given a compact hypersurface with nowhere zero Gauss-Kronecker in a space form, if for each point of $M$ and every direction $Y_p$ there exists a corresponding shadow boundary making a constant angle with respect to $Y$, then $M$ must be totally umbilical.

## 2 Preliminaries

In this section we fix our notation. Our ambient space $(N^{n+1}, \langle\ ,\ \rangle)$ will be a Riemannian manifold with connection $D$.

**Definition 2.1.** Let $M$ be an immersed submanifold of $N$, and let $Y : N \to TN$ be a vector field in $N$. The *shadow boundary* of $M$ with respect to $Y$ is the following subset of $M$:

$$S\partial(M, Y) = \{p \in M \mid Y_p \in T_pM\}. \tag{1}$$

In [2], J. Choe gave the above definition of shadow boundary of Riemannian submanifolds, calling it horizon. Using the generalized Morse index theorem, he related this concept with the index of stability of a complete minimal surface in $\mathbb{R}^3$.

In this paper we will work with Riemannian manifolds which admit a closed conformal vector field $Y$.

**Definition 2.2.** We say that $Y$ is a closed conformal vector field if there exist $\varphi : N \to \mathbb{R}$ smooth such that for every vector field $X$ in $N$ we have

$$D_X Y = \varphi X,$$

where $D$ is the Levi-Civita connection of $N$.

For example, if $Y$ is parallel we can take $\varphi = 0$. In the Euclidean space $\mathbb{R}^{n+1}$, if $Y$ is a radial vector field, the corresponding $\varphi$ is constant equal to one. This means that in particular, our results hold for constant (i.e., parallel) and radial vector fields in $\mathbb{R}^{n+1}$.

Hereafter $Y$ will denote a closed conformal vector field in $N$. We will suppose that $Y$ does not vanish on the submanifold $M$.

# 3 Closed conformal vector fields and shadow boundaries

In general a shadow boundary is just a closed subset of $M$. The next result says that the shadow boundary is a smooth submanifold of $M$ in a region when the Gauss-Kronecker curvature is different from zero.

**Proposition 3.1.** *Let $M$ be an oriented immersed hypersurface in $N$ and $Y$ a closed conformal vector field in $N$. Let $p$ be a point in $S\partial(M, Y)$ where either of the following conditions hold:*

- *The shape operator satisfies $A(Y_p) \neq 0$; or*
- *The Gauss-Kronecker curvature of $M$ is not zero at $p$.*

*Then there exists a neighbourhood $U$ of $p$ in $M$ such that $S\partial(M, Y) \cap U$ is a hypersurface of $M$.*

*Proof.* Let $\xi$ be a unit vector field which is everywhere normal to $M$. Recall that the Gauss-Kronecker curvature is given by $\det A$, where $A(X) = -D_X\xi$ is the shape operator of $M$ relative to $\xi$. Note that if $\det A_p \neq 0$, then $A_p$ is a linear isomorphism and then $A(Y_p) \neq 0$, so we just analyze this case.

Let $U$ be a neighbourhood of $p$ where the following conditions hold for each point $q$ in $U$:

- $Y_q$ is not orthogonal to $M$, i.e. $Y_q^T \neq 0$.

- $A(Y_q^T) \neq 0$; here $Y^T$ denotes the projection of $Y$ into $TM$.

It is clear that the above neighbourhood $U$ exists. Let us define the smooth function $F : U \to \mathbb{R}$ by $F = \langle Y, \xi \rangle$. Therefore, $S\partial(M, Y) \cap U = F^{-1}(0)$. We will prove that zero is a regular value of $F$. If $X$ denotes a vector field tangent to $M$ and $\nabla F$ denotes the gradient of $F$, then

$$\begin{aligned} \langle \nabla F, X \rangle &= XF = \langle D_X Y, \xi \rangle + \langle Y, D_X \xi \rangle \\ &= \langle \varphi X, \xi \rangle - \langle Y^T, A(X) \rangle = -\langle A(Y^T), X \rangle. \end{aligned}$$

Note that the above implies that $\nabla F = -A(Y^T)$. By our assumptions, $A(Y^T) \neq 0$ and therefore $\nabla F$ does not vanish in $U$. In particular, 0 is a regular value of $F$ and therefore $S\partial(M, Y) \cap U$ is a hypersurface. □

We now express the properties of the shadow boundary in terms of the second fundamental form of $M$.

**Lemma 3.2.** *Let $M$ be an immersed hypersurface in $N$ with second fundamental form $\alpha$ and $Y$ be a closed conformal vector field in $N$. If $p \in L = S\partial(M, Y)$ and $A(Y_p) \neq 0$, then $\alpha(Y_p, X_p) = 0$ for every $X_p \in T_pL$.*

*Proof.* By Proposition 3.1, we now that under the given hypotheses $L$ is a hypersurface of $N$, at least in a neighbourhood of $p$. Let $\xi$ be a local unit normal vector field defined in such a neighbourhood of $p$ and $\gamma$ a smooth curve in $L$ such that $\gamma(0) = p$ and $\gamma'(0) = X_p$. Since every point of $\gamma$ belongs to $L$ we have $\langle Y, \xi \rangle = 0$; taking the derivative with respect to $X$ we obtain

$$0 = \langle D_X Y, \xi \rangle + \langle Y, D_X \xi \rangle = \langle \varphi X, \xi \rangle - \langle Y, A(X) \rangle = -\langle Y, A(X) \rangle$$

as in the proof of Proposition 3.1. Since $Y_p = Y_p^T$, we have

$$0 = -\langle Y_p, A(X_p) \rangle = -\langle A(Y_p), X_p \rangle = -\langle \alpha(Y_p, X_p), \xi \rangle;$$

the above implies that $\alpha(Y_p, X_p) = 0$. □

**Definition 3.3.** Let $M$ be an immersed hypersurface in $N$. Given a tangent vector $v_p \in T_pM \setminus \{0\}$, we say that a shadow boundary $S\partial(M, Y)$ is *generated* by $v_p$ if $Y$ is a closed conformal vector field satisfying the initial condition $Y_p = v_p$.

As an important example, we analyze shadow boundaries of hypersurfaces in space forms in the next Proposition.

**Proposition 3.4.** *Let $N$ be a space form $\mathbb{Q}_c^{n+1}$. Given an immersed hypersurface $M$ of $\mathbb{Q}_c^{n+1}$ and any vector $v_p \in T_pM \setminus \{0\}$, there exists a shadow boundary $S\partial(M, Y)$ generated by $v_p$.*

*Proof.* Let $M$ and $v_p \in T_pM$ be as in the statement. Define a point $p_0 \in M$ by $p_0 = \exp_p(-v_p)$ when $c \leq 0$ and by $p_0 = \exp_p(-\lambda v_p)$ for a suitable $\lambda > 0$ such that $\lambda |v_p| < \pi/\sqrt{c}$, the diameter of $\mathbb{Q}_c^{n+1}$, if $c > 0$.

Consider the gradient vector field $\nabla d$ of the distance function $d(\cdot, p_0)$. As usual, $\nabla d$ is defined only in $\mathbb{Q}_c^{n+1} \setminus \{p_0\}$ (or $\mathbb{Q}_c^{n+1} \setminus \{p_0, -p_0\}$ for $c > 0$). In this domain, following [1], p. 205, we define the position vector $Y$ in $\mathbb{Q}_c^{n+1}$ relative to $p_0$ by

$$Y_q = S_c(d(q, p_0)) \nabla d_q,$$

where

$$S_c(s) = \begin{cases} s, & c = 0; \\ \sin(s\sqrt{c})/\sqrt{c}, & c > 0; \\ \sinh(s\sqrt{-c})/\sqrt{-c}, & c < 0. \end{cases}$$

$Y$ is a closed conformal vector field. In fact, it is well known that the gradient of a distance function satisfies $D_{\nabla d} \nabla d = 0$, from which we obtain

$$D_{\nabla d} Y = S_c' \nabla d, \quad S_c' = \frac{dS_c}{ds};$$

on the other hand, in [1], p. 207, it is proved (see equation (2.2) there) that

$$D_X Y = S_c' X$$

for every vector field $X$ transversal to $\nabla d$. The last two equations imply that $Y$ is closed conformal.

From the very definition of $p_0$ we have that $Y_p$ is a scalar multiple of $v_p$, so that by multiplying by a suitable constant we obtain a closed conformal vector field assuming the vaue $v_p$ at $p$. □

**Proposition 3.5.** *Let $M$ be an immersed hypersurface in $N$ with second fundamental form $\alpha$. Let $p \in M$ be a point where the Gauss-Kronecker curvature of $M$ is different from zero. Then for every $n$-dimensional subspace $V$ of $T_pM$ there exists a vector $v_p \in T_pM$ such that the tangent space at $p$ of every shadow boundary $S\partial(M, Y)$ generated by $v_p$ is equal to $V$.*

*Proof.* Let $\xi$ be a local normal unit vector field of $M$ around $p$ and $A$ its associated shape operator. Since $\det A_p \neq 0$, $A : T_pM \to T_pM$ is invertible. Thus we may take $v_p \in T_pM$ such that $A(v_p)$ spans $V^\perp$, the orthogonal complement of $V$ in $T_pM$.

Let $Y$ be any closed conformal vector field which takes the value $v_p$ at $p$. By Proposition 3.1 we know that the shadow boundary $L := S\partial(M, Y)$ is an embedded hypersurface of $M$ around $p$.

By Lemma 3.2, $\alpha(Y_p, X_p) = 0$ for every $X_p \in T_pL$. Therefore,

$$0 = \langle \alpha(Y_p, X_p), \xi \rangle = \langle A(Y_p), X_p \rangle = \langle A(v_p), X_p \rangle.$$

Since $A(v_p)$ spans $V^\perp$, the above proves that $T_pL = V$. □

Let us recall that a point $p$ in a hypersurface $M \subset N$ is an umbilic point of $M$ if and only if every tangent vector to $M$ at $p$ is a principal direction of the shape operator $A$ of $M$ at $p$. As noted in the abstract, we may use shadow boundaries in order to detect umbilic points in a hypersurface. The following results show how this can be done.

**Definition 3.6.** *A submanifold $L$ of a hypersurface $M \subset N$ is* invariant *under the shape operator $A$ of $M$ if for every point $p \in L$, we have that $A(T_pL) \subset T_pL$.*

**Proposition 3.7.** *Let $M$ be an immersed hypersurface in $N$. Let $p \in M$ be a point where the Gauss-Kronecker curvature is different from zero, and such that for every $v_p \in T_pM \setminus \{0\}$ there exists a shadow boundary $S\partial(M, Y)$ generated by $v_p$ which is invariant under the shape operator of $M$. Then $p$ is an umbilic point of $M$.*

*Proof.* Since the Gauss-Kronecker curvature of $M$ does not vanish at $p$, the shape operator $A$ of $M$ relative to a unit normal vector field $\xi$ is invertible.

Fix a vector $v_p$ and let $Y$ be as in the hypotheses. We will show that $L = S\partial(M, Y)$ is orthogonal to $Y$ at $p$. By Lemma 3.2, $\alpha(Y_p, X_p) = 0$ for every $X_p \in T_pL$. Since $A$ is invertible and $L$ is invariant under $A$, then $A(T_pL) = T_pL$. So, for every $Z_p \in T_pL$ there exists $X_p \in T_pL$ such that $Z_p = A(X_p)$. Therefore,

$$\langle Z_p, Y_p\rangle = \langle A(X_p), Y_p\rangle = \langle \alpha(Y_p, X_p), \xi\rangle = 0.$$

Since $v_p$ was arbitrarily chosen, by Corollary 4.2 we conclude that $p$ is an umbilic point of $M$. □

We have the following straightforward application of Proposition 3.7

**Corollary 3.8.** *Given a surface $M$ with nowhere zero Gauss-Kronecker curvature in a three dimensional Riemannian manifold $N$, if for every $p \in M$ and every $v_p \in T_pM \setminus \{0\}$ there exists a shadow boundary $S\partial(M, Y)$ generated by $v_p$ which is a line of curvature of $M$, then $M$ is totally umbilical in $N$.*

# 4 Totally umbilical hypersurfaces in space forms

In this section we will apply our previous results in order to characterize the totally umbilical hypersurfaces of the space forms $\mathbb{Q}_c^{n+1}$ in terms of shadow boundaries.

**Proposition 4.1.** *Let $M$ be an immersed hypersurface in $\mathbb{Q}_c^{n+1}$. Let $p \in M$ be any point where the Gauss-Kronecker curvature of $M$ is different from zero. A vector $v_p \in T_pM \setminus \{0\}$ determines a principal direction of $M$ if and only if there exists a shadow boundary $S\partial(M, Y)$, generated by $v_p$, which is orthogonal to $Y$ at $p$.*

*Proof.* Let us assume first that the shadow boundary $S\partial(M,Y)$ generated by $v_p$ is orthogonal to the closed conformal vector field $Y$at $p$. By Lemma 3.2, $\langle A(Y_p), X_p\rangle = \langle \alpha(Y_p, X_p), \xi\rangle = 0$ for every $X_p \in T_pL$. This says that $A(Y_p)$ is orthogonal to $T_pL$. But $Y_p$ is also orthogonal to $T_pL$; therefore, $A(Y_p)$ is a multiple of $Y_p$; since $Y_p = v_p$, this means that $v_p$ determines a principal direction.

Conversely, let us assume that a vector $v_p$ determines a principal direction, so that $A(v_p) = \lambda v_p$. Note that since $\det A_p \neq 0$, we have $\lambda \neq 0$.

By Proposition 3.4, we may construct a closed conformal vector field $Y$ such that $Y_p = v_p$. By Lemma 3.2, we have that

$$\lambda\langle Y_p, X_p\rangle = \langle \lambda Y_p, X_p\rangle = \langle A(Y_p), X_p\rangle = \langle \alpha(Y_p, X_p), \xi\rangle = 0,$$

where $X_p \in T_pL$. This proves that $Y_p = v_p$ is orthogonal to the shadow boundary $L$ generated by $v_p$. □

The following result is an immediate consequence of Proposition 4.1.

**Corollary 4.2.** *Let $M$ be an immersed hypersurface in $\mathbb{Q}_c^{n+1}$. Let $p \in M$ be any point where the Gauss-Kronecker curvature of $M$ is non zero. The point $p$ is an umbilic point of $M$ if and only if for every $v_p \in T_pM\setminus\{0\}$, there exists a shadow boundary $S\partial(M,Y)$, generated by $v_p$, which is orthogonal to $Y$ at $p$.*

In order to prove our main results, we apply Proposition 2 and Remark 2 in [5], which describe the structure of a complete manifold possessing a globally defined closed conformal vector field. For completeness we rephrase here the facts relevant in our setting.

**Proposition 4.3** (Montiel)**.** *Let $Y$ be a non-trivial closed conformal vector field defined globally in the space form $\mathbb{Q}_c^{n+1}$. Then $Y$ has at most two zeroes and*

1. *If $Y$ has exactly one zero $q$, then $\mathbb{Q}_c^{n+1}\setminus\{q\}$ is isometric to a warped product $\mathbb{R}^+ \times_f \mathbb{S}^n$. If $(r,p) \in \mathbb{R}^+ \times \mathbb{S}^n$ represent the polar coordinates of a point, then $Y_{(r,p)} = f(r)p$. The spheres $\{r\}\times\mathbb{S}^n$ give a foliation of $\mathbb{Q}_c^{n+1}\setminus\{q\}$.*
2. *If $Y$ has two zeroes $q, -q$, then $\mathbb{Q}_c^{n+1}\setminus\{q,-q\}$ is isometric to a warped product $(0,\pi)\times_f\mathbb{S}^n$. The spheres $\{r\}\times\mathbb{S}^n$ give a foliation of $\mathbb{Q}_c^{n+1}\setminus\{q-q\}$.*
3. *If $Y$ has no zeroes, then $\mathbb{Q}_c^{n+1}$ is isometric to a warped product $\mathbb{R}\times_f\mathbb{Q}_d^n$, where $\mathbb{Q}_d^n$ is a space form of curvature $d$. In this case, $Y = f(r)\partial_r$ and the slices $\{r\}\times\mathbb{Q}_d^n$ foliate $\mathbb{Q}_c^{n+1}$.*

We say that a regular curve $L$ in a surface $M \subset \mathbb{Q}_c^3$ contains a principal direction of $M$ at $p \in L$ if the tangent line of $L$ at $p$ is generated by a principal direction of the shape operator of $M$ in $\mathbb{Q}_c^3$.

**Proposition 4.4.** *Let $M$ be an immersed surface in $\mathbb{Q}_c^3$. Then every compact regular shadow boundary $L := S\partial(M, Y)$ of $M$ relative to a globally defined closed conformal vector field $Y$ contains at least two principal directions of $M$ at two different points of $L$. In particular, if $M$ is a compact surface with nowhere zero Gaussian curvature then every shadow boundary of $M$ contains at least two principal directions of $M$ at two different points.*

*Proof.* By Proposition 4.1, we have to prove the existence of two points in $L$ where $L$ is orthogonal to $Y$.

Suppose the first case in Proposition 4.3 holds; that is, suppose that $Y$ has exactly one zero $q$ and that $\mathbb{Q}_c^{n+1} \setminus \{q\} = \mathbb{R}^+ \times_f \mathbb{S}^n$. Since $M$ is compact and $L$ is a closed subset of $M$, there are real numbers $r_1, r_2$ such that $L$ is contained in a slab $[r_1, r_2] \times \mathbb{S}^n$. If $[r_1, r_2]$ is the smallest interval with this property, then each sphere $\{r_i\} \times \mathbb{S}^n$ is tangent to the shadow boundary at their contact points; since the spheres are orthogonal to $Y$, the same happens to the shadow boundary; that is, the shadow boundary is orthogonal to $Y$ at its contact points with the mentioned spheres.

A completely similar argument holds for the cases where $Y$ has two or no zeroes. □

Finally, we will prove our main result.

**Theorem 4.5.** *Let $M$ be a compact hypersurface with nowhere zero Gauss-Kronecker curvature in $\mathbb{Q}_c^{n+1}$. If for every $p \in M$ and every direction $Y_p \in T_pM \setminus \{0\}$ there exists a shadow boundary $S\partial(M, Y)$ generated by $Y_p$ which makes a constant angle with respect to the globally defined closed conformal vector field $Y$, then $M$ is totally umbilical.*

*Proof.* By Proposition 3.1, every shadow boundary is a hypersurface of $M$. Moreover, since $M$ is compact and each shadow boundary $L = S\partial(M, Y)$ is closed in $M$, we have that each $L$ is compact. The idea of the proof is to use compactness to prove that the constant angle between $S\partial(M, Y)$ and $Y$ should be $\pi/2$, i.e., that every shadow boundary $S\partial(M, Y)$ is orthogonal to $Y$.

The argument here is analogous to that of Proposition 4.4. Let $p$ be any point of $M$, $Y_p \in T_pM$ be any non zero tangent vector and $Y$ the corresponding closed conformal vector field.

By Proposition 4.3, the region where $Y$ has no zeroes has a decomposition by slices $\{r\} \times \mathbb{Q}_d^n$, each one orthogonal to $Y$. By compactness, the shadow boundary $L$ is tangent to one of these slices at, say, a point $p$. Therefore, the angle between $Y_p$ and $T_pS\partial(M, Y)$ is $\pi/2$. Now we apply Corollary 4.2 to conclude that the point $p$ is an umbilic point. □

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