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Stability of Non Liquid Bridges

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To Professor Gervasio Colares on the occasion of his eightieth birthday

Abstract

We give a new, simpler proof characterizing the stable non liquid bridges. Numerical examples are given which show that the conditions imposed on the functionals in these theorems are essential. We also show that arbitrary convex non liquid bridges are always stable.

1 Introduction

If a volume of liquid is placed between two horizontal planes, it will form a capillary surface. In order to reach the least energy configuration, the surface of the drop will have constant mean curvature and will meet each of the planes in a right angle. Strictly speaking, for this to be true, the volume of the drop should be sufficiently small (how small depends on the drop's composition via the Bond number) so that gravity can be neglected. Furthermore, we have only spoken so far about a first order criteria. In order for the drop to actually form, it must, in addition, be stable. By this we mean that the second variation of

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area is non negative for all volume preserving variations. An important result, shown independently by Athanssenas, [1], and Vogel, [8], states that the only stable capillary surfaces which lie between the two planes are spherical caps and sufficiently short cylinders. (See also an additional proof obtained in [7] which applies when the ambient dimension is at less than or equal to eight).

In [3] the result of [1] and [8] was extended to solutions of the same volume constrained free boundary problem with the area replaced by a rotationally invariant anisotropic surface energy. The critical surfaces, called anisotropic capillary surfaces, have constant anisotropic mean curvature in their interior.

The aims of this paper are to

- Give a new and far simpler proof of our result characterizing the stability of non liquid bridges.
- Give numerical results which clarify the role played by the curvature condition on the Wulff shape in these results.
- Show that arbitrary convex non liquid bridges are stable.

In the anisotropic case, a Maximum Principle again applies with the consequence that any anisotropic capillary surface embedded between the two planes is necessarily axially symmetric. In a previous paper we constructed all axially symmetric solutions. Here again, we give a much simpler proof of one of the results (Theorem 3.1) of [3]. Our proof avoids the complicated eigenvalue analysis used before and demonstrates explicitly the role played by a hypothesis on the curvature of the Wulff shape which was needed to obtain this result. We conclude the paper by giving graphical evidence which indicates that this curvature hypothesis is, in fact, essential.

From a physical perspective, our result might be applied to study the equilibrium shape of a drop nematic liquid crystal trapped between two horizontal planes. In many applications, the director field is aligned by applying and external force field. If this force fixes the director to be orthogonal to the boundary planes (homeotropic alignment), our result indicates that above a critical volume, only cylinders occur as stable equilibria. If the director is instead fixed to be parallel to the planes (parallel alignment), then the numerical results given below indicate that anisotropic unduloids could occur as stable equilibria for volumes below a critical level. Above this level, cylinders again occur as the unique stable equilibria.

Physical experiments involving nematic liquid crystal bridges are described in [6]. The bridges that form for sufficiently large volumes appear to be cylinders. However, in these experiments no alignment of the director field was made. Since we expect the same outcome when the director field is aligned, a careful experiment which tests our result physically appears to be difficult.

Finally, we wish to point out that with the results of our recent paper, [5], many of the results of this paper extend to capillary surfaces for a much larger class of functionals. These functionals have a Wulff shape which is of "product form", where the circular orbit of a point is replaced by an arbitrary convex curve.

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2 Preliminaries

Let γ be a positive C^∞ function on the unit sphere S^2. For an immersion $X : \Sigma \to \mathbf{R}^3$ of two dimensional orientable C^∞ maniford Σ to the three dimensional Euclidean space $\mathbf{R}^3$ with Gauss map $\nu = (\nu_1, \nu_2, \nu_3) : \Sigma \to S^2 \subset \mathbf{R}^3$, we define a parametric elliptic functional

$$\mathcal{F}[X] := \int_\Sigma \gamma(\nu)\, d\Sigma\,.$$

The anisotropic energy density, γ , is assumed to satisfy a convexity condition so that the surface

$$\chi := \gamma\nu + D\gamma : S^2 \to \mathbf{R}^3$$

is a smooth, convex surface with strictly positive curvature. This surface is called the *Wulff shape* and it is the absolute minimizer of the free energy $\mathcal{F}$ among all surfaces enclosing the same volume.

The anisotropic mean curvature Λ is defined by

$$\Lambda = -(\mathrm{div}\, D\gamma - 2H\gamma),$$

where H is the mean curvature of the surface. The volume constrained critical surfaces for the functional $\mathcal{F}$ are characterized by the constancy of the anisotropic mean curvature. Because of the convexity condition, the equation Λ = constant is absolutely elliptic and one can conclude that a Maximum Principle analogous to the one for CMC surfaces applies.

Assume that γ is rotationally symmetric. Then the Wulff shape W is a surface of revolution. Surfaces with constant anisotropic mean curvature (CAMC surfaces) which are rotationally symmetric with the same rotation axis as that of W are called *anisotropic Delaunay surfaces.* We will recall a simple parameterization of these surfaces. Details can be found in [2]. For convenience, we will write $\mathbf{R}^3 \approx \mathbf{C} \times \mathbf{R}$. We parameterize a surface of revolution Σ by

$$X(s,\theta) = (x(s)e^{i\theta}, z(s)) \ ,$$

where s is the arc length parameter for the generating curve $(x(s), z(s))$. In particular, we parameterize the axially symmetric Wulff shape W as $\chi(\sigma,\theta) = (u(\sigma)e^{i\theta}, v(\sigma))$, $\sigma \in [0,\pi]$, $\theta \in [0, 2\pi]$. It is understood that the points on Σ correspond to points on W where the normals to the two surfaces agree, i.e. via the Gauss map. Under this correspondence, the coordinates of an ansisotropic Delaunay surface satisfy

$$2ux + \Lambda x^2 = c \ , \qquad c \in \mathbf{R}, \tag{2.1}$$

$$z = \int x_u \, dv \ . \tag{2.2}$$

It is understood that the equation (2.1) is solved for x in terms of u and this function is then used in (2.2). The constant c is a flux parameter and Λ is the anisotropic mean curvature. We will always normalize the orientation so that $\Lambda \leq 0$ holds.

Let $\bar{u}$ denote the maximum value of u on W. The six possible types of anisotropic Delaunay surfaces corrsepond to

- Planes, $\Lambda = c = 0$.
- Anisotropic catenoids, $\Lambda = 0$, $c \neq 0$.
- Rescalings of the Wulff shape, $\Lambda < 0$, $c = 0$.
- Cylinders, $\Lambda < 0$, $\bar{u}^2 + \Lambda c = 0$.
- Anisotropic unduloids, $\Lambda < 0$, $c > 0$, $\bar{u}^2 + \Lambda c > 0$.
- Anisotropic nodoids, $\Lambda < 0$, $c < 0$.

Anisotropic unduloids and nodoids are periodic surfaces. The anisotropic unduloids are embedded but nodoids have self intersections. We refer to the maximum and minimum distance from a point on the surface to the rotation axis as a bulge (B) and a neck (N), respectively.

We let μ_1 (resp. μ_2) denote the principal curvature of the Wulff shape W along a meridian (resp. parallel). The μ_i's are functions of ν_3. The μ_i's are also the reciprocals of the eigenvalues of $(D^2\gamma + \gamma I)_\nu$. The orientation of W is the inward one so $\mu_i > 0$ holds. If X is a surface of revolution with the same rotational symmetry as W, the anisotropic principal curvatures of X are respectively

$$\lambda_1 := k_1/\mu_1 = \frac{\Lambda}{2} + \frac{c}{2x^2}\,, \quad \lambda_2 := k_2/\mu_2 = \frac{\Lambda}{2} - \frac{c}{2x^2}\,, \tag{2.3}$$

where k_1 (resp. k_2) are the principal curvature of X along a meridian (resp. parallel) with respect to the outward pointing normal.

3 A Free boundary problem

Assume that γ is rotationally symmetric: $\gamma = \gamma(\nu_3)$. Set

$$\Pi_0 := \{x_3 = 0\}, \quad \Pi_1 := \{x_3 = h\}, \quad (h > 0), \qquad \Omega := \{0 \le x_3 \le h\}.$$

For an oriented embedded surface $X : (\Sigma^2, \partial\Sigma) \to (\Omega, \Pi_0 \cup \Pi_1)$, we define an energy:

$$\mathcal{E}[X] := \mathcal{F}[X] + \omega_0\mathcal{A}_0[X] + \omega_1\mathcal{A}_1[X],$$

where $\mathcal{A}_i[X]$ is the area in Π_i which is bounded by $C_i := X(\partial\Sigma) \cap \Pi_i$, and ω_i are constants. $\omega_i\mathcal{A}_i[X]$ is called a wetting energy.

We define the volume $V[X]$ of X as the usual volume of the three-dimensional domain enclosed by $X(\Sigma) \cup \Pi_1 \cup \Pi_2$. A variation $X_t : (\Sigma, \partial\Sigma) \to (\Omega, \partial\Omega)$ of X will be called an *admissible variation* if $V[X_t] = V[X]$ for all t.

For $p \in \Sigma$, there exists a uniquely determined point $\bar{G}(p)$ in the Wulff shape W such that the normal $\nu(p)$ of X coincides with the outward pointing unit normal to W at $\bar{G}(p)$. We call this map $\bar{G} : \Sigma \to W$ the anisotropic Gauss map of X.

If X is a critical point of $\mathcal{E}$ for all admissible variations, we call X an *anisotropic capillary surface*.

Proposition 3.1 ([4]). *An embedding $X : (\Sigma, \partial\Sigma) \to (\Omega, \partial\Omega)$ is an anisotropic capillary surface if and only if there holds: $\Lambda \equiv \Lambda_0$ in Σ for some constant Λ_0, and*

$$\langle \overline{G}, E_3 \rangle \equiv -(-1)^i \omega_i \quad \text{on } C_i \tag{3.1}$$

($i = 0, 1$), that is, the contact angle of X with each Π_i is a constant.

By using the maximum principle, we can show the following:

Corollary 3.1 ([4]). *An anisotropic capillary surface $X : (\Sigma, \partial\Sigma) \to (\Omega, \partial\Omega)$ for $\gamma = \gamma(\nu_3)$ is a CAMC surface of revolution with vertical rotation axis, and its genus is zero.*

In view of Corollary 3.1, an anisotropic capillary surface $X : (\Sigma, \partial\Sigma) \to (\Omega, \Pi_0 \cup \Pi_1)$ is represented as

$$X(s, \theta) = (x(s)\cos\theta, x(s)\sin\theta, z(s)), \quad s_1 \le s \le s_2.$$

Let $X_t = X + (\xi + \psi\nu)t + \mathcal{O}(t^2)$ be an admissible variation of X, where ξ is the tangential component of the variation vector field. Then, the second variation of energy is

$$\delta^2 \mathcal{E} := \left. \frac{d^2 \mathcal{E}(X_t)}{dt^2} \right|_{t=0} = -\int_\Sigma \psi J[\psi]\, d\Sigma + \oint_{\partial\Sigma} \psi B[\psi]\, d\tilde{s} =: \mathcal{I}[\psi], \tag{3.2}$$

where J is the self-adjoint Jacobi operator

$$\begin{aligned} J[\psi] &= \operatorname{div}(A\nabla\psi) + \langle A d\nu, d\nu \rangle \psi \\ &= x^{-1}\{(\mu_1^{-1} x \psi_s)_s + \mu_2^{-1} x^{-1} \psi_{\theta\theta}\} \\ &+ \{\mu_1^{-1}(x'' z' - x' z'')^2 + \mu_2^{-1}(x^{-1} z')^2\}\psi, \end{aligned} \tag{3.3}$$

$$A := (D^2\gamma + \gamma 1)|_\nu,$$

and

$$\begin{aligned} B[\psi] &= \begin{cases} \langle A\nabla\psi + \psi(\nu_3/n_3) A\, d\nu(n), n \rangle, & n_3 \ne 0, \\ \psi, & n_3 = 0, \end{cases} \\ &= \begin{cases} -(-1)^i \mu_1^{-1}\{\psi_s - (z''/z')\psi\}, & z' \ne 0, \\ \psi, & z' = 0 \end{cases} \end{aligned}$$

on the boundary C_i. Here $n = (n_1, n_2, n_3)$ is the outward pointing normal along $\partial\Sigma$. We also remark that, the first variation $\delta\Lambda$ of the anisotropic mean curvature satisfies

$$\delta\Lambda = J[\psi] + \langle \nabla\Lambda, \xi \rangle \tag{3.4}$$

([2]).

Definition 3.1. An anisotropic capillary surface is said to be *stable* if $\delta^2\mathcal{E} \geq 0$ for all admissible variations, otherwise it is said to be unstable.

The analytic condition for stability is

Proposition 3.2. *An anisotropic capillary surface $X : (\Sigma, \partial\Sigma) \to (\Omega, \partial\Omega)$ is stable if and only if $\mathcal{I}[\psi] \geq 0$ holds for all C^∞ functions ψ on Σ which satisfy*

(i) $\psi(w) = 0$ for $w \in \partial\Sigma$ where X is tangent to $\partial\Omega$, and (ii) $\int_\Sigma \psi \, d\Sigma = 0$.

Now we consider two eigenvalue problems which are associated with the second variation of the energy $\mathcal{E}$ for admissible variations.

Denote by $\lambda_i^0 := \lambda_i^0(\Sigma)$ the ith eigenvalue of the following eigenvalue problem:

$$J[\psi] = -\lambda\psi \text{ in } \Sigma, \quad B[\psi]|_{\partial\Sigma} = 0, \qquad \psi \in H^1([s_1, s_2] \times S^1) = H^1(\Sigma). \tag{3.5}$$

Also, denote by $\lambda_i := \lambda_i([s_1, s_2])$ the ith eigenvalue of the following eigenvalue problem:

$$J[\varphi] = -\lambda\varphi \text{ in } [s_1, s_2], \quad B[\varphi]|_{\partial[s_1,s_2]} = 0, \qquad \varphi \in H^1([s_1, s_2]), \tag{3.6}$$

where $J[\varphi]$, $B[\varphi]$ mean $J[\phi]$, $B[\phi]$ for $\phi(s, \theta) = \varphi(s)$ $(\forall\theta)$, respectively. The latter eigenvalue problem is associated with the second variation of the energy $\mathcal{E}$ for rotationally symmetric admissible variations.

If $z'(s) \neq 0$ for all $s \in (s_1, s_2)$, then X is stable if and only if X is stable for rotationally symmetric admissible variations ([4]).

Because of the volume constraint, the stability of X cannot be determined by using only the eigenvalues. However, one has the following result:

Proposition 3.3 ([4])**.** *Assume that $z'(s) \neq 0$ for any $s \in (s_1, s_2)$.*

(i) If $\lambda_1 \geq 0$, then X is stable.

(ii) Assume that $\lambda_1 < 0 \le \lambda_2$ *holds. (ii-1) If there exists a solution* $\varphi : [s_1, s_2] \to \mathbf{R}$ *of* $J[\varphi] = 1$ *on* $[s_1, s_2]$ *with* $B[\varphi] = 0$ *on* $\partial[s_1, s_2]$*, then* X *is stable if and only if*

$$\int_\Sigma \varphi \, d\Sigma \ge 0 \, .$$

(ii-2) If no solution of the problem exists, then the surface is unstable.

(iii) If $\lambda_2 < 0$*, then* X *is unstable.*

3.1 Neutral wetting

In this subsection we consider the case $\omega_0 = \omega_1 = 0$. It was shown in [3] that any anisotropic capillary surface for this problem is necessarily an anisotropic Delaunay surface which meets the supporting planes in right angles (cf. Proposition 3.1 and Corollary 3.1).

Theorem 3.1 ([3])**.** *Assume that the Wulff shape* W *is a surface of revolution. We assume, without loss of generality, that the bulge is situated on the plane* $x_3 = 0$ *. Also assume that the curvature of the generating curve of* W *are non decreasing functions of arc length as measured in the vertical direction from the plane* $x_3 = 0$*. Then any half period of an anisotropic unduloid (bulge to neck) is unstable for the free boundary problem.*

Before giving a new proof of Theorem 3.1, we will give some remarks. From the theorem and simple calculations, it is easy to see the following result which was also proved in [3]:

Corollary 3.2 ([3])**.** *The only stable anisotropic capillary surfaces which join the two planes are cylinders whose height* h *and volume* V *satisfy*

$$\frac{\pi}{h^3} - \frac{\mu_1(0)}{\mu_2(0)V} \ge 0 \, . \tag{3.7}$$

Proof of Theorem 3.1. For a half period of an anisotropic unduloid (bulge to neck), we will construct a piecewise smooth, volume-preserving variation which diminishes the energy $\mathcal{F}$.

Starting with any sufficiently smooth surface $X : \Sigma \to \mathbf{R}^3$, we define an energy for maps $f : \Sigma \to S^2$, by

$$E_\nu[f] := \sum_{j=1..3} \int_\Sigma (D^2\gamma + \gamma I)_\nu \nabla f_j \cdot \nabla f_j \, d\Sigma =: \int_\Sigma (D^2\gamma + \gamma I)_\nu \nabla f \cdot \nabla f \, d\Sigma \, .$$

Note that to define this energy, one needs to have the immersion X in order to identify $T_p\Sigma$ wth $T_{\nu(p)}S^2$ so as to be able to apply $(D^2\gamma + \gamma I)_\nu$ to the tangent vectors ∇f_j. It was shown in [2] that if X has constant anisotropic mean curvature, then the Gauss map ν is a critical point of this energy among all maps having the same boundary values, i.e. $\delta E_\nu[\nu] = 0$. The variable ν which appears as a subscript should be regarded as being fixed in this problem.

Let $X(s,\theta) = (x(s)e^{i\theta}, z(s))$ be a half period of an anisotropic unduloid (bulge to neck). Represent the Gauss map of X as

$$\nu(s,\theta) = ((\cos\alpha(s))e^{i\theta}, \sin\alpha(s)).$$

We compute

$$d\nu = ((-\sin\alpha)e^{i\theta}, \cos\alpha)\alpha_s ds + i\cos\alpha(e^{i\theta}, 0)d\theta\ .$$

Since the first term on the right is parallel to the tangential part of $E_3 := (0,0,1)$ and the second term is orthogonal to the same vector, we obtain.

$$Ad\nu = ((-\sin\alpha)e^{i\theta}, \cos\alpha)\frac{\alpha_s}{\mu_1}ds + i\frac{\cos\alpha}{\mu_2}(e^{i\theta}, 0)d\theta\ .$$

From this we get that the energy of the Gauss map for ν is

$$E = 2\pi\int\left(\frac{\alpha_s^2}{\mu_1} + \frac{\cos^2\alpha}{x^2\mu_2}\right)x\, ds = 2\pi\int\left(\alpha_t^2 + \frac{\cos^2\alpha}{K_W}\right)dt\ ,$$

where $t_s := \mu_1/x$, and $K_W = \mu_1\mu_2$ (> 0) is the Gauss curvature of W. In the variational problem for the energy, the quantities, K_W and μ_j are not varied, they are considered as coefficients. The Euler-Lagrange equation is the *anisotropic pendulum equation* (A. P. E.)

$$\alpha_{tt} + \frac{\cos\alpha\sin\alpha}{K_W} = 0\ . \tag{3.8}$$

For any solution, we have a first integral

$$\alpha_t^2 + \int\frac{\sin 2\alpha}{K_W}\, d\alpha = \text{constant.}$$

Assume that the parameters have been chosen such that $s = 0 \leftrightarrow t = 0 \leftrightarrow \alpha = 0$ and that $s = 0$ corresponds to a bulge. After integrating by parts, the previous equation can be written as

$$\alpha_t^2 = \frac{\cos^2\alpha}{K_W} - \int_0^\alpha(\cos^2\alpha)\partial_\alpha(1/K_W)\, d\alpha + b\,, \tag{3.9}$$

where $b := (\alpha_t^2 - (\cos^2\alpha)/K_W)_{t=0}$. The linearized A. P. E. is

$$\mathcal{L}_1[\psi] = \psi_{tt} + \left(\frac{\cos 2\alpha}{K_W} + \frac{1}{2}(\sin 2\alpha)\partial_\alpha(1/K_W)\right)\psi = 0\,.$$

If α solves the A. P. E., then, function α_t is a solution of the linearized equation as can be verified by differentiation. The numerical value of this function is $\alpha_t = -xk_1/\mu_1$.

The second variation of the anisotropic energy $\mathcal{F}$ for volume-preserving variation $X_\epsilon = X + \epsilon\psi\nu + \mathcal{O}(\epsilon^2)$ is (cf. [2], Proposition 2.2)

$$\begin{aligned}\delta^2\mathcal{F} &= \int_\Sigma (D^2\gamma + \gamma I)_\nu \nabla\psi\cdot\nabla\psi - \langle (D^2\gamma+\gamma I)d\nu, d\nu\rangle \psi^2\, d\Sigma \\ &= 2\pi\int \psi_t^2 - (\alpha_t^2 + \frac{\cos^2\alpha}{K_W})\psi^2\, dt,\end{aligned}$$

provided ψ_t vanishes on the boundary. Substituting using (3.9), we obtain

$$\delta^2\mathcal{F} = 2\pi\int \psi_t^2 - \left(\frac{2\cos^2\alpha}{K_W} - \int_0^\alpha (\cos^2\alpha)\partial_\alpha(1/K_W)\,d\alpha + b\right)\psi^2\, dt\,.$$

We now assume that K_W is increasing on $[0,\pi/2]$. This holds in our case. Then $\partial_\alpha(1/K_W) < 0$ holds, and hence the energy form corresponding to the linearized A. P. E. is

$$\begin{aligned} D &:= \int \psi_t^2 - \left(\mathcal{L}_1[\psi] - \psi_{tt}\right)\psi\, dt \\ &= \int \psi_t^2 - \left(\frac{\cos 2\alpha}{K_W} + \frac{1}{2}(\sin 2\alpha)\partial_\alpha(1/K_W)\right)\psi^2\, dt \\ &\geq \int \psi_t^2 - \left(\frac{\cos 2\alpha}{K_W}\right)\psi^2\, dt \\ &= \int \psi_t^2 - \left(\frac{2\cos^2\alpha}{K_W} - \frac{1}{K_W}\right)\psi^2\, dt \\ &= \frac{1}{2\pi}\delta^2\mathcal{F} + \int\left(\alpha_t^2 + \frac{1-\cos^2\alpha}{K_W}\right)\psi^2\, dt \\ &> \frac{1}{2\pi}\delta^2\mathcal{F}, \end{aligned} \tag{3.10}$$

since $(\sin 2\alpha)\partial_\alpha(1/K_W) \leq 0$ holds for all $\alpha \in [0,\pi/2]$.

It follows that if $D \leq 0$ holds on any interval, then $\delta^2\mathcal{F} \leq 0$ holds on that interval also, provided that $\psi\psi_t$ vanishes at both endpoints. Recall $\alpha_t = -xk_1/\mu_1$, and set

$$\alpha_t^+ := \{\max \alpha_t, 0\}, \quad \alpha_t^- := \{-\max\alpha_t, 0\}.$$

We take $\psi := p\alpha_t^+ - \alpha_t^-$ where p is a positive constant chosen so that

$$\int_\Sigma \psi \, d\Sigma = 0$$

holds. Such a choice of constant is possible since, the curvature k_1 of the generating curve Γ of any half period of an anisotropic unduloid (bulge to neck) changes sign. Let I denote the circle of inflection points of Γ. They divide the surface into two parts; bulge to inflection point $=: BI$ and $IN :=$ inflection point to neck. BI, (respectively IN) is exactly the set of points where $k_1 < 0$, (respectively $k_1 > 0$) holds for outward pointing normal. Note that $\psi \geq 0$ holds in BI and $\psi \leq 0$ holds in IN.

There exists a piecewise smooth, rotationally symmetric variation $X_\epsilon = X + \epsilon\psi\nu + \mathcal{O}(\epsilon^2)$ which preserves the volume and has singularities only on I, and satisfies $X_\epsilon = X$ on I. Recall $\mathcal{L}_1[\alpha_t] = 0$. Also, note that $\alpha_{tt} = -\cos\alpha \sin\alpha / K_W$ from (3.8). Hence, the second variation $\delta^2\mathcal{F}$ of the energy $\mathcal{F}$ satisfies, by using (3.10),

$$\begin{aligned} \delta^2\mathcal{F} &< 2\pi \int_B^N \psi_t^2 - (\mathcal{L}_1[\psi] - \psi_{tt})\psi \, dt \\ &= 2\pi \int_B^N \psi_t^2 + \psi_{tt}\psi \, dt = 2\pi \big\{ (\psi\psi_t)_B - (\psi\psi_t)_N \big\} = 0. \end{aligned}$$

Hence, X_ϵ diminishes the energy $\mathcal{F}$. Therefore, there exists a smooth function φ on Σ which satisfies $\int_\Sigma \varphi \, d\Sigma = 0$ and $\mathcal{I}[\varphi] < 0$. This implies that X is unstable. **q.e.d**

Figures 1 and 2 shows the results of two energy minimzation simulations. For Figure 1, the energy density is $\gamma = 1 - 0.45\nu_3^2$. For the corresponding functional, the curvature hypothesis of the previous theorem does not hold. The initial curve is a cylinder with volume 0.64π. The volume of the surfaces generated by the evolving curves as well as the heights are preserved. The resulting minimization terminates in an anisotropic unduloid (the blue curve). This indicates that the hypothesis on the curvature cannot be omitted.

In Figure 2, the functional is given by $\gamma = 1 + 0.45\nu_3^2$. For the corresponding functional, the curvature condition holds. The initial curve is now chosen to be an an unduloid. Again, the simulation preserves both the enclosed volume and the height. In this case, as predicted, the surfaces terminate in a cylinder.

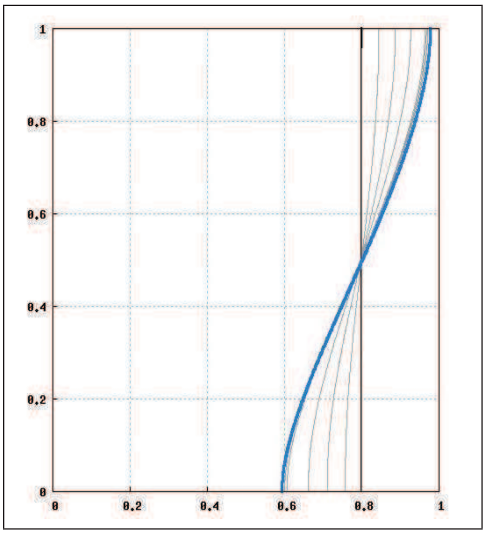

Figure 1: $\gamma = 1 - 0.45\nu_3^2$. Energy minimization simulation which terminates in an anisotropic unduloid.

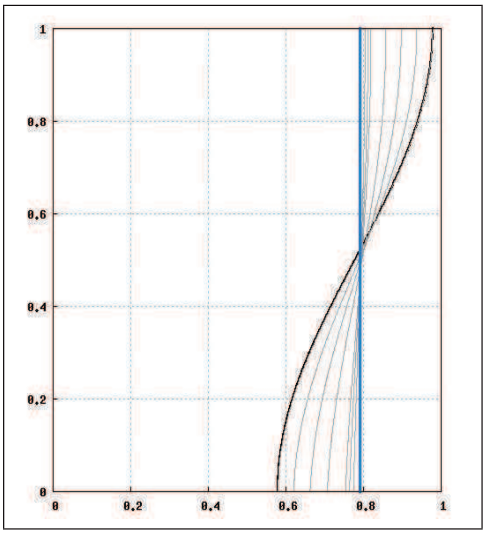

Figure 2: $\gamma = 1 + 0.45\nu_3^2$. Energy minimization simulation, which terminates in a cylinder.

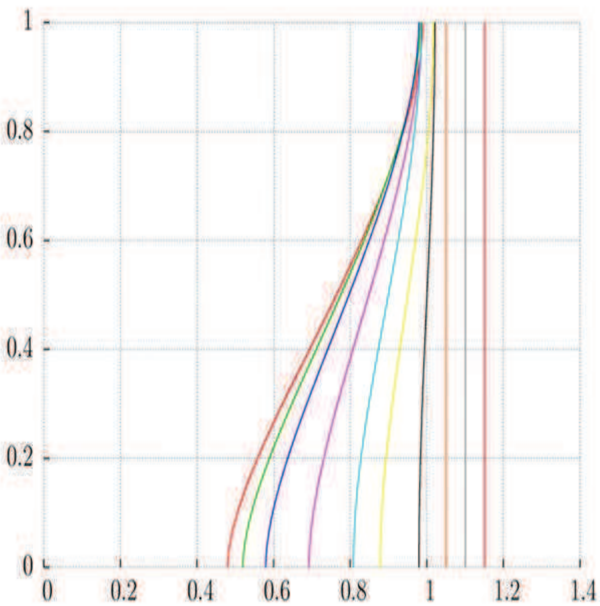

Figure 3: $\gamma = 1 - 0.45\nu_3^2$. Generating curves of least energy surfaces

Figure 3 shows a variety of curves which generate least energy surfaces. The functional is again the one having density $\gamma = 1 - 0.45\nu_3^2$. Note that for large volumes, the least energy surface is a cylinder while for small volumes, unduloids may occur.

3.2 Lyophobic wetting

In this subsection we consider the free boundary problem with lyophobic wetting energy on the two free boundaries. The total energy is

$$\mathcal{E} = \mathcal{F} + \omega_0 \mathcal{A}_0 + \omega_1 \mathcal{A}_1 \,, \tag{3.11}$$

where ω_i are nonnegative constants, and $(\omega_0, \omega_1) \neq (0, 0)$. Again, it was shown in [4] that any anisotropic capillary surface for this problem is necessarily an anisotropic Delaunay surface (cf. Corollary 3.1). In [4], we proved that, in the case where $\omega_0 = \omega_1 > 0$ holds, under the assumption on the curvature of the Wulff shape given in Theorem 3.1, only convex parts of anisotropic Delaunay surfaces are stable. Here we will prove that, for any nonnegative ω_i's, any convex parts of anisotropic Delaunay surfaces are stable also. Moreover, we will give a certain sufficient condition for the instability in this case.

An idea developed by Vogel for constant mean curvature surfaces, is that in place of solving the problem for φ in Proposition 3.3 (ii), one can produce a one parameter family of rotationally symmetric anisotropic capillary surfaces which, in our case, depend on the anisotropic mean curvature $X(\Lambda) = X_0 + (\Lambda - \Lambda_0)\delta X + \mathcal{O}((\Lambda - \Lambda_0)^2)$ with all surfaces having their boundaries lie on the supporting planes and which meet the supporting planes in a prescribed angle which is independent of Λ, where Λ_0 is the anisotropic mean curvature of the initial surface X_0. Let $V = V(\Lambda)$ denote the enclosed volume, then

$$\frac{dV}{d\Lambda} = \int_\Sigma \phi \, d\Sigma \,, \tag{3.12}$$

where $\phi := \delta X \cdot \nu$. Observe, that since the deformation preserves the contact angle, ϕ satisfies the boundary condition $B[\phi] = 0$ on Π_i. From (3.4),

$$J[\phi] = \frac{d(\Lambda - \Lambda_0)}{d\Lambda} = 1$$

holds. Therefore by (3.12), the immersion will be stable, (resp. unstable), if $dV/d\Lambda \geq 0$, (resp. < 0) holds.

Lemma 3.1. *Assume that $\omega_0, \omega_1 \geq 0$ hold. Let X be a convex anisotropic capillary surface. If the generating curve of X has inflection points at both end points, then the second eigenvalue λ_2 of the problem (3.6) is zero. Otherwise, $\lambda_1 < 0 < \lambda_2$ holds.*

Proof. The former half is a part of Lemma 8.3 in [4]. The latter half is a part of Lemma 7.4 in [4]. **q.e.d.**

Theorem 3.2. *Assume that $\omega_0, \omega_1 > 0$ holds. Then any anisotropic capillary surface whose generating curve contains two or more inflection points in its interior is unstable. (We do not assume here the curvature of W satisfies the condition of Theorem 3.1).*

Proof. First consider the two subregions U_i' of the surface bounded by a bulge β and one of the boundary curves C_i, $i = 0, 1$. Consider the eigenvalue problem $(J + \mu)f = 0$ in U_i', $f = 0$ on β, $B[f] = 0$ on C_i. The eigenvalue μ_1 is given by

$$\mu_1 = \inf \frac{-\int_{U'} fJ[f]\, d\Sigma + \oint_{C_i} fB[f]\, dL}{\int_{U'} f^2\, d\Sigma}\,, \tag{3.13}$$

where the infimum is over all smooth functions with $f \equiv 0$ on β.

Note that ν_3 satisfies $J[\nu_3] = 0$, $\nu_3 = 0$ on β, and, on C_i,

$$\nu_3 B[\nu_3] = \nu_3 \frac{(-1)^i}{\mu_1 z'} k_1\,.$$

Since $k_1 > 0$ holds at both endpoints, this quantity is negative, so using $f = \nu_3$ in (3.13), we get $\mu_1 < 0$.

Now let ψ satisfy $(J + \lambda_2)\psi = 0$ in Σ with $B[\psi] = 0$ on $\partial\Sigma$. Again, the zero set of ψ is a circle $\alpha \subset \Sigma$. We can assume that α lies below the bulge of Σ. Let U be the domain in Σ bounded by α and C_1. Consider the eigenvalue problem $(J + \mu)f = 0$ in U with $B[f] = 0$ on C_1, $f = 0$ on α. Then $\mu_1(U) = \lambda_2$. However it follows from the variational characterization of μ_1, that $\mu_1(U) < \mu_1(U_1') < 0$ holds. This shows that $\lambda_2 < 0$ holds from which the instability easily follows. **q.e.d.**

Theorem 3.3. *Let Σ be a convex anisotropic capillary surface for (3.11) with $\omega_i \geq 0$, $i = 0, 1$. Then Σ is stable. (We do not assume here the curvature of W satisfies the condition of Theorem 3.1).*

Proof. Let Σ be a convex anisotropic capillary surface. Then Σ is either part of the Wulff shape or it is contained in the convex part of an anisotropic unduloid or nodoid. If it is part of the Wulff shape it is automatically stable (in fact it is energy minimizing by Winterbottom's theorem [9]. See also [4]). We can therefore ignore this case. In the other cases, from (2.1), we can write

$$x = \frac{u + \sqrt{u^2 - a}}{-\Lambda} \,, \tag{3.14}$$

where $a := -\Lambda c$ and $a > 0$ (resp. < 0) corresponds to an unduloid (resp. nodoid). The branch of the solution of (2.1) with the minus sign corresponds to the negatively curved parts of the respective surfaces. In (3.14), we always have $u^2 > a$ in the interior of Σ. On each boundary C_i, the case $u^2 = a$ occurs if and only if the surface is an unduloid with an inflection point on C_i.

The idea is to produce a one parameter family of anisotropic capillary surfaces Σ_Λ parameterized by their anisotropic mean curvatures all having the same height. If $\Sigma = \Sigma_{\Lambda_0}$, then assuming that $\lambda_1 < 0 \leq \lambda_2$ holds, the necessary and sufficient condition for stability is

$$\frac{dV}{d\Lambda}(\Lambda_0) \geq 0 \,. \tag{3.15}$$

Because of (3.1) and (3.14), the height is given by

$$h = h(a, \Lambda) = \int_{-\omega_0}^{\omega_1} dz = \int_{-\omega_0}^{\omega_1} x_u \, dv = \frac{1}{-\Lambda} \int_{-\omega_0}^{\omega_1} \frac{u + \sqrt{u^2 - a}}{\sqrt{u^2 - a}} \, dv \,,$$

while the volume is

$$\begin{aligned} V =& V(a, \Lambda) = \pi \int_{-\omega_0}^{\omega_1} x^2 \, dz = \pi \int_{-\omega_0}^{\omega_1} x^2 x_u \, dv \\ =& \frac{\pi}{(-\Lambda)^3} \int_{-\omega_0}^{\omega_1} \frac{(u + \sqrt{u^2 - a})^3}{\sqrt{u^2 - a}} \, dv \,. \end{aligned}$$

Straightforward calculation yields

$$\frac{\partial h}{\partial a} = \frac{1}{2(-\Lambda)} \int_{-\omega_0}^{\omega_1} \frac{u}{(u^2 - a)^{3/2}} \, dv > 0 \,,$$

therefore we can regard a as varying with Λ in such a way that the height is fixed, i.e. $h(a(\Lambda), \Lambda) \equiv$ constant. We can differentiate this equation implicitly with respect to Λ to obtain,

$$\begin{aligned} 0 = \frac{dh}{d\Lambda} &= \frac{\partial h}{\partial \Lambda} + \frac{\partial h}{\partial a} \frac{da}{d\Lambda} \\ &= \frac{h}{(-\Lambda)} + \frac{a_\Lambda}{2(-\Lambda)} \int_{-\omega_0}^{\omega_1} \frac{u}{(u^2 - a)^{3/2}} \, dv \,. \end{aligned} \tag{3.16}$$

Thus,

$$a_\Lambda = \frac{-h_\Lambda}{h_a} = 2h\left(\int_{-\omega_0}^{\omega_1} \frac{u}{(u^2-a)^{3/2}}\,dv\right)^{-1}. \tag{3.17}$$

We then differentiate V with respect to Λ, to obtain

$$\frac{dV}{d\Lambda} = V_\Lambda + V_a a_\Lambda = \frac{V_\Lambda h_a - h_\Lambda V_a}{h_a}\,.$$

Since we have already shown that the denominator is positive, we see by direct computation that stability is equivalent to the inequality

$$\begin{aligned} 3 & \int_{-\omega_0}^{\omega_1} \frac{(u+\sqrt{u^2-a})^3}{\sqrt{u^2-a}}\,dv \int_{-\omega_0}^{\omega_1} \frac{u}{(u^2-a)^{3/2}}\,dv \\ \geq & \int_{-\omega_0}^{\omega_1} \frac{u+\sqrt{u^2-a}}{\sqrt{u^2-a}}\,dv \times \int_{-\omega_0}^{\omega_1} \frac{(u+\sqrt{u^2-a})^2(-2\sqrt{u^2-a}+u)}{(u^2-a)^{3/2}}\,dv\,. \end{aligned}$$

This is not difficult to verify. Begin by writing

$$\begin{aligned} (u+\sqrt{u^2-a})^2(-2\sqrt{u^2-a}+u) &= (u+\sqrt{u^2-a})(u+\sqrt{u^2-a})\,\cdot \\ &\cdot\ \left([-\sqrt{u^2-a}+u] - \sqrt{u^2-a}\right) \\ &= a(u+\sqrt{u^2-a}) \\ &- \sqrt{u^2-a}(u+\sqrt{u^2-a})^2\,. \end{aligned}$$

By replacing the factor above containing this term and rearranging, we see that the previous inequality holds if and only if

$$\begin{aligned} & 3\int_{-\omega_0}^{\omega_1} \frac{(u+\sqrt{u^2-a})^3}{\sqrt{u^2-a}}\,dv \int_{-\omega_0}^{\omega_1} \frac{u}{(u^2-a)^{3/2}}\,dv \\ +& \int_{-\omega_0}^{\omega_1} \frac{(u+\sqrt{u^2-a})}{\sqrt{u^2-a}}\,dv \int_{-\omega_0}^{\omega_1} \frac{(u+\sqrt{u^2-a})^2}{u^2-a}\,dv \\ \geq & \int_{-\omega_0}^{\omega_1} \frac{(u+\sqrt{u^2-a})}{\sqrt{u^2-a}}\,dv \left(\int_{-\omega_0}^{\omega_1} \frac{au}{(u^2-a)^{3/2}}\,dv + \int_{-\omega_0}^{\omega_1} \frac{a}{u^2-a}\,dv\right) \\ =& \; a\int_{-\omega_0}^{\omega_1} \frac{(u+\sqrt{u^2-a})}{\sqrt{u^2-a}}\,dv \int_{-\omega_0}^{\omega_1} \frac{u}{(u^2-a)^{3/2}}\,dv \\ +& \int_{-\omega_0}^{\omega_1} \frac{(u+\sqrt{u^2-a})}{\sqrt{u^2-a}}\,dv \int_{-\omega_0}^{\omega_1} \frac{a}{u^2-a}\,dv\,. \end{aligned}$$

This inequality then follows from the fact that $(u+\sqrt{u^2-a})^2 \geq a$. **q.e.d.**

Figure 4 shows a variety of curves which generate least energy surfaces. The functional is again the one having density $\gamma = 1 - 0.45\nu_3^2$. There is a

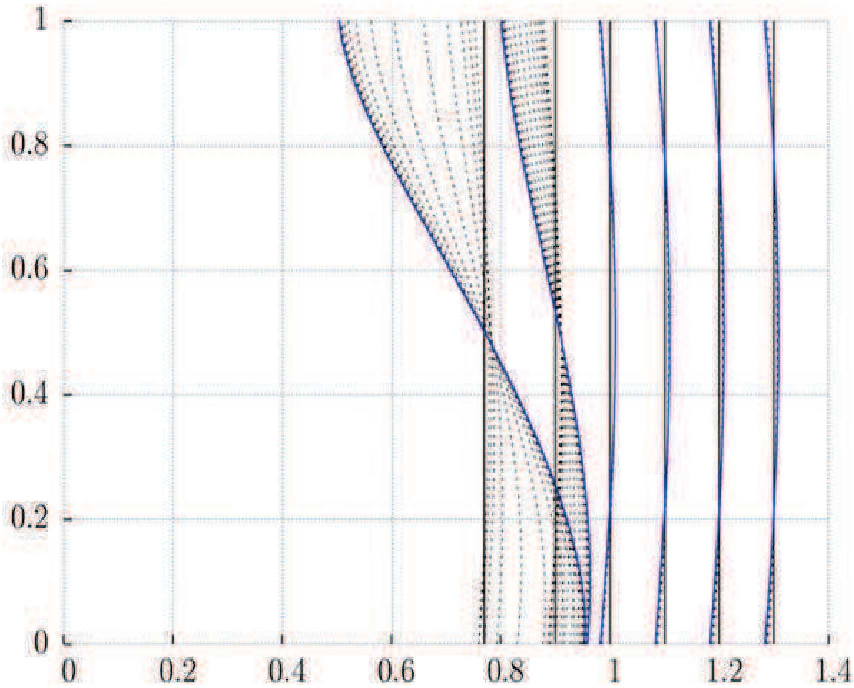

Figure 4: The functional is again $\gamma = 1 - 0.45\nu_3^2$, $\omega = 0.001$. One sees the initial (black) curve which determines the volume and the corresponding final (blue) curve which generates an unduloid. For large volumes, the unduloids are convex but for smaller volumes there are unduloids whose generating curve contains inflection points.

wetting energy contributed from both planes with $\omega_0 = \omega_1 = 0.001$. Although the curves for larger volumes appear to generate cylinders, they in fact generate convex parts of anisotropic unduloids. For smaller volumes, it appears that stable unduloids with one inflection point can form. This is in contrast to the situation when the curvature condition holds, when, for $\omega_0 = \omega_1$, only convex equilibria are stable.

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