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The $r-$mean curvature equation of a graph and scalar flat hypersurfaces revisited

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Dedicated to Professor Gervásio Colares on his 80th birthday

Abstract

Among the results we discuss in this work we will see how to transform non-singular analytic curves Σ in $\mathbb{C}^2$ into strictly convex scalar flat 3-dimensional hypersurfaces.

1 Introduction.

A number of authors have studied the existence of hypersurfaces with prescribed curvatures in $\mathbb{R}^{n+1}$. As we know, such a hypersurface is locally given as a graph Γ_f of a real-valued function f defined over a domain $\Omega \subset \mathbb{R}^n$. In the induced metric the first and the second fundamental form of Γ_f are given respectively by

$$\begin{cases} g_{ij} = \delta_{ij} + f_i f_j \\ h_{ij} = -f_{ij}/W \end{cases}$$

where $W = \sqrt{1+|\nabla f|^2}$, $\nabla f = (f_1, \cdots, f_n)$ and $(f_{ij}) \equiv f_{**}$ is the Hessian of f. The $r-$th mean curvature H_r of Γ_f vanishes if and only if

$$\sum_{i_1<\cdots<i_r} D_{i_1\cdots i_n}(f) = 0, \tag{1.1}$$

where $D_{i_1\cdots i_n}(f)$ is the determinant of the $n\times n$ matrix obtained from $G = (g_{ij})$ by replacing its $i_1, \cdots, i_r$ columns by the corresponding columns of f_{**}. In

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particular, Γ_f is scalar flat if and only if

$$\sum_{i<j} D_{ij}(f) = 0.$$

According to [8], the partial differential equation $H_r(f) = 0$ is elliptic precisely at the points where $\text{rank}(f_{**}) \geq r$. Obviously, the ellipticity of $H_r(f) = 0$ holds if Γ_f is a strictly convex hypersurface. In this particular case

$$H_r(f) = 0, \quad \det f_{**} \neq 0. \tag{1.2}$$

Even though the elipticity of equation $H_2(f) = 0$ fails for flat solutions, M. L. Leite [13] was able to proof the following interesting result.

Theorem 1.1. *If the graph of $f : \mathbb{R}^3 \to \mathbb{R}$ is flat at infinity and $H_2 \equiv 0$, then Γ_f is globally flat.*

Remark 1. In [13], M. L. Leite extended the above result to the case $H_2 \geq 0$, proving the veracity of Geroch conjecture [5] in this special case.

We also note that a solution of equation (1.2) on a bounded domain with smooth boundary is completely determined by the values of f and ∇f on the boundary. This uniqueness result is a consequence of the following more general Fact (proved in section 5).

Theorem 1.2. *Let $M_1, M_2 \subset \mathbb{R}^{n+1}$ be strictly convex compact hypersurfaces with the same smooth boundary X. If*

a) for some r, the r-th mean curvatures of M_1 and M_2 vanishes.

b) X have the same induced orientations and the same normal vectors.

Then $M_1 = M_2$.

The main part of this paper is dedicated to the construction of non trivial examples of strictly convex scalar flat hypersurfaces of $\mathbb{R}^4$. This was discussed in a previous paper [1] and is accomplished by looking at the structure of the focal locus of a complex analytic curve in complex euclidian 2-space $\mathbb{C}^2$. This construction lead us to the following unexpected result (see proof in section 4).

Theorem 1.3. *The focal locus of a non-singular analytic curve* Σ *in* $\mathbb{C}^2$ *is the union*

$$F(\Sigma) = F_\Sigma \cup \Sigma^*$$

of a singular set Σ^* *and a strictly convex scalar flat hypersurface* F_Σ *of* $\mathbb{R}^4$*.*

Remark 2*.* We may as well think of Theorem 1.3 as a transform, i.e., from a complex analytic curve we construct its focal locus that in turn produce a solution of the equation

$$\sum_{i<j} D_{ij}(f) = 0.$$

2 The mean curvature equations of a graph

This section is concerned with real-valued functions $f : \Omega \to \mathbb{R}$ defined over a domain $\Omega \subset \mathbb{R}^n$. In the induced metric the first and the second fundamental form of Γ_f are given respectively by

$$\begin{cases} g_{ij} = \delta_{ij} + f_i f_j \\ h_{ij} = -f_{ij}/W \end{cases}$$

where $W = \sqrt{1 + |\nabla f|^2}$, $\nabla f = (f_1, \cdots, f_n)$ and $(f_{ij}) \equiv f_{**}$ is the Hessian of f. A straightforward computation shows that $\det G = W^2$.

Let $D_{i_1 \cdots i_n}(f)$ be the determinant of the $n \times n$ matrix obtained from $G = (g_{ij})$ by replacing its $i_1, \cdots, i_r$ columns by the corresponding columns of f_{**}. In this section we always assume that f is a solution of the partial differential equation

$$\epsilon_r(f) =: \sum_{i_1 < \cdots < i_r} D_{i_1 \cdots i_n}(f) = 0, \quad \det f_{**} \neq 0. \tag{2.1}$$

Proposition 2.1. *The function f is a solution of the equation (6.1) if and only if Γ_f is a strictly convex hypersurface with $H_r \equiv 0$.*

Proof. The proof is a consequence of the following lemma. □

Lemma 1. *Let $f : \Omega \to \mathbb{R}$ be a real function defined over a domain $\Omega \subset \mathbb{R}^n$. Then*

$$(-W)^{2+r} \tbinom{n}{r} H_r = \epsilon_r(f). \tag{2.2}$$

Proof. Let $k_1, k_2, \cdots, k_n$ be the principal curvatures of Γ_f. They are the roots of the polynomial equation $p(\lambda) = 0$, where

$$\begin{aligned} \det G\, p(\lambda) &= \det(\ \lambda G + f_{**}/W\ ) \\ &= \textstyle\sum_\sigma sgn(\sigma) \prod_{i=1}^n \left(\lambda g_{i\sigma(i)} - h_{i\sigma(i)}\right) \\ &= \textstyle\sum_\sigma sgn(\sigma)\, g_{1\sigma(1)} \cdots g_{n\sigma(n)} \prod_{i=1}^n \left(\lambda - \frac{h_{i\sigma(i)}}{g_{i\sigma(i)}}\right) \\ &= \textstyle\sum_{r=0}^n (-1/W)^r \binom{n}{r} \epsilon_r(f) \lambda^{n-r}. \end{aligned}$$

In the above identities σ is a permutation of $\{1, \cdots, n\}$ while $sgn(\sigma)$ denotes the sign of the permutation σ. Since

$$p(\lambda) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} H_r \lambda^{n-r},$$

the result follows by comparing the coefficients of $p(\lambda)$ in the two expressions.

□

Corollary 2.2. *Let $A_{ij}(f)$ be the i, j cofactor of f_{**}. Then Γ_f is a $3-$dimensional scalar flat hypersurface of $\mathbb{R}^4$ if and only if*

$$\epsilon_2(f) = \sum_{i<j} g_{ij} A_{ij}(f) = 0.$$

As we know a $3-$dimensional hypersurface $M \subset \mathbb{R}^4$ is scalar flat if and only $H_2 = 0$. In this particular case, if M is a graph of a function f, then f is a solution of the partial differential equation

$$\epsilon_2(f) = A(f) + 2B(f) = 0,$$

where

$$\begin{cases} A(f) = g_{11}\left(f_{22}f_{33} - f_{23}^2\right) + g_{22}\left(f_{11}f_{33} - f_{13}^2\right) + g_{33}\left(f_{11}f_{22} - f_{12}^2\right) \\ B(f) = g_{12}\left(f_{13}f_{23} - f_{12}f_{33}\right) + g_{13}\left(f_{12}f_{32} - f_{13}f_{22}\right) + g_{23}\left(f_{21}f_{31} - f_{23}f_{11}\right). \end{cases}$$

This lengthy but highly symmetric equation was studied by M. L. Leite [13] who among other things proved the so called Geroch's Conjecture for a smooth graph in $\mathbb{R}^4$. Explicitly, she proved the following result.

Theorem 2.3. *If the graph of $f : \mathbb{R}^3 \to \mathbb{R}$ is flat at infinity and $\epsilon_2(f) \geq 0$, then Γ_f is globally flat.*

3 The focal locus construction

In this section we will describe a method for constructing a special class of scalar flat hypersurfaces in the Euclidean space $\mathbb{R}^4$.

3.1 Basic definitions

Let $\Sigma \subset \mathbb{C}^2$ be a non-singular holomorphic curve. We will denote by $\langle , \rangle$ the standard inner product on $\mathbb{C}^2$ and by $J : \mathbb{C}^2 \to \mathbb{C}^2$ the multiplication by $\sqrt{-1}$. If $z \in \mathbb{C}^2$, we set

$$|z| = \langle z, z\rangle^{1/2}.$$

Let ∇ be the Riemannian connection on $\mathbb{C}^2$. The second fundamental form of Σ is defined by

$$B_{V,W} \equiv (\nabla_V W)^{N,} \tag{3.1}$$

for $V, W \in T_X\Sigma \equiv$ tangent space of Σ at X. Here $(\quad)^N$ denotes projection onto $N_X\Sigma \equiv$ normal space of Σ at X. Given a normal vector $\xi_X \in N_X\Sigma$ we define $A^\xi : T_X\Sigma \to T_X\Sigma$ by

$$A^\xi(V) = -(\nabla_V \xi)^T, \tag{3.2}$$

where ξ is an arbitrary vector field in $\mathbb{C}^2$ with the property that ξ is normal to Σ in a neighborhood of X and $(\quad)^T$ denotes projection onto $T_X\Sigma$.

Remark 3. It is a well known fact (cf. [11]) that the $N(\Sigma)$-valued bilinear form B is symmetric and also complex bilinear, i.e., $B_{JV,W} = JB_{V,JW} = B_{V,JW}$. Note that A and B are related by

$$< B_{V,W}, \xi >=< A_\xi(V), W > . \tag{3.3}$$

In particular A^ξ is self-adjoint. The eigenvalues $\pm\lambda(X, \xi)$ of A^ξ are independent of the choice of ξ at X, and if $B \neq 0$, they vanish only at isolated points. In this paper we will avoid those points.

Given a normal vector field ξ of unit length at X on Σ we associate to ξ the eigendirection of A^ξ with positive eigenvalue λ. There are two eigenvectors of unit length on this "eingen-line", υ_ξ and $-\upsilon_\xi$. We denote by ξ_t the unit normal vector $\xi cost + (J\xi)sint$. It follows from the above remark that the eigenvalues of A^{ξ_t} do not depend on t. They are given by λ and $-\lambda$ and the eigenline corresponding to $-\lambda$ is determined by $J\upsilon_{\xi_t}$. An easy computation shows that

$$\pm\upsilon_{\xi_t} = \upsilon_\xi \cos(t/2) + J\upsilon_\xi \sin(t/2).$$

From now on we will choose the sign of υ_ξ so that

$$\upsilon_{\xi_t} = e^{it/2}\upsilon_\xi, \quad i = \sqrt{-1}. \tag{3.4}$$

Definition 1. The focal locus of Σ is the set

$$F_\Sigma = \{X + \rho(X)\xi \ : \ X \in \Sigma, \ \ \xi \in N_1\Sigma\},$$

where $N_1\Sigma$ is the unit normal sphere bundle of Σ and $\rho(X) \equiv 1/\lambda(X,\xi)$.

In order to determine the structure of the focal locus we will consider the mapping $l : \Sigma \times \mathbb{S}^1 \to F_\Sigma \subset \mathbb{C}^2$ given by

$$l(X,t) = X + \rho(X)e^{it}\nu_X, \tag{3.5}$$

where ν is a unit normal vector field on Σ. One can prove easily that at a point $(X,t) \in \Sigma \times \mathbb{S}^1$ we have

$$l_*\upsilon_\nu \wedge l_*J\upsilon_\nu \wedge l_*\partial/\partial t = 2\rho(\upsilon_{\nu_t}.\rho)\nu_t \wedge J\upsilon_{\nu_t} \wedge J\nu_t. \tag{3.6}$$

Lemma 2. *The mapping* $l : \Sigma \times \mathbb{S}^1 \to \mathbb{C}^2$ *given by (3.5) is an immersion at* $(X,t) \in \Sigma \times S^1$ *if and only if*

$$< \nabla\rho, \upsilon_{\nu_t} > \neq 0. \tag{3.7}$$

Proof. Lemma 2 follows from equation (3.6). □

From now on we will assume that Σ contains no critical points of ρ. In particular $|\nabla\rho| \neq 0$ and we can define the vector fields υ_1, υ_2 on Σ by

$$J\upsilon_1 = \upsilon_2 = \nabla\rho/|\nabla\rho|. \tag{3.8}$$

Remark 4. The vector field ν in (3.5) may be chosen in such a way that $\upsilon_\nu = \upsilon_1$. This vector field is obviously unique. With this notation we have the following result.

Lemma 3. *The focal locus of* Σ *is the union* $F_\Sigma \cup \Sigma^*$ *of a 3-dimensional manifold* F_Σ *and a singular set* Σ^**. Moreover*

$$F_\Sigma = \{X + \rho(X)e^{it}\nu_X : X \in \Sigma, \ 0 < t < 2\pi\}$$

$$\Sigma^* = \{X + \rho(X)\nu_X : X \in \Sigma\},$$

where ν is the unique unit normal vector field on Σ such that $\upsilon_\nu = \upsilon_1$.

Proof. A point $X^* \in F_\Sigma$ may be written as $X^* = l(X,t)$ for some $X \in \Sigma$ and $t \in [0, 2\pi)$. We observe now that $\langle \nabla\rho, \upsilon_{\nu_t}\rangle = |\nabla\rho| \sin(t/2) > 0$. The result follows by applying Lemma 2. □

3.2 The second fundamental form of F_Σ

In this section we analyse the geometric structure on the focal locus of a non-singular analytic curve Σ in $\mathbb{C}^2$. Over F_Σ we define a field of orthonormal frames $X^* e_1, e_2, e_3, e_4$ such that for $X^* = X + \rho(X)\xi \in F_\Sigma$ we have

$$e_1 = J\upsilon_\xi, \quad e_2 = \xi, \quad e_3 = J\xi, \quad e_4 = \upsilon_\xi. \tag{3.9}$$

The vector field e_4 is obviously normal to F_Σ. We let ω_A , $1 \le A \le 4$, be the dual coframe of e_A. To e_A we also associate the connection 1-forms ω_{AB} given by

$$de_A = \sum_B \omega_{AB} e_B. \tag{3.10}$$

The Cartan structure equations are

$$\begin{cases} d\omega_A = \sum_B \omega_{AB} \wedge \omega_B \\ d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB}, \quad \omega_{AB} + \omega_{BA} = 0. \end{cases} \tag{3.11}$$

Let $T(F_\Sigma)$ and $T^*(F_\Sigma)$ be respectively the tangent and cotangent bundle of F_Σ. The second fundamental form II of F_Σ is a section on $T^*(F_\Sigma) \otimes T(F_\Sigma)$ whose components with respect to the given orthonormal frame e_A are

$$II = (h_{ij}), \quad \omega_{i4} = \sum_{j=1}^{3} h_{ij}\omega_j. \tag{3.12}$$

Lemma 4. *At the point $X^* = X + \rho(X)\xi_X \in F_\Sigma$ we have*

$$(\upsilon_\xi.\rho^2) II = \begin{bmatrix} -|\nabla\rho|^2/2 & (J\upsilon_\xi).\rho & -\upsilon_\xi.\rho \\ (J\upsilon_\xi).\rho & -2 & 0 \\ -\upsilon_\xi.\rho & 0 & 0 \end{bmatrix}. \tag{3.13}$$

Proof. We are going to use the moving frame method. For this we consider the distinguished orthonormal frame field υ_A on Σ obtained by making

$$\upsilon_2 = J\upsilon_1 = \nabla\rho/|\nabla\rho|, \quad \upsilon_3 = \nu, \quad \upsilon_4 = J\nu \tag{3.14}$$

where ν is the unique normal vector field such that $\upsilon_\nu = \upsilon_1$. We then associate to υ_A its dual coframe $\theta_A, 1 \leq A \leq 4$ and denote by θ_{AB} the 1-forms on Σ given by

$$d\upsilon_A = \sum_{B=1}^{4} \theta_{AB}\upsilon_B. \tag{3.15}$$

We recall that the focal locus is given by the mapping $l : \Sigma \times \mathbb{S}^1 \to \mathbb{C}^2$ where

$$l(X,t) = X + \rho e^{it}\upsilon_3. \tag{3.16}$$

Taking the differential of (3.16) gives

$$dl = dX + d\rho e^{it}\upsilon_3 + \rho dt e^{it}\upsilon_4 + \rho e^{it} d\upsilon_3.$$

By construction

$$\begin{cases} \rho\theta_{31} = \rho\theta_{42} = -\theta_1 \\ \rho\theta_{32} = \rho\theta_{14} = \theta_2. \end{cases} \tag{3.17}$$

Therefore

$$dl = (1 - e^{it})\theta_1\upsilon_1 + (1 + e^{it})\theta_2\upsilon_2 + d\rho e^{it}\upsilon_3 + \rho e^{it}[dt + \theta_{34}]\upsilon_4.$$

Then

$$dl = 2\left[-\sin\left(\frac{t}{2}\right)\theta_1 + \cos\left(\frac{t}{2}\right)\theta_2\right]e_1 + d\rho e_2 + \rho[dt + \theta_{34}]e_3. \tag{3.18}$$

It follows that

$$\begin{cases} l^*\omega_1 = 2\left[-\sin\left(\frac{t}{2}\right)\theta_1 + \cos\left(\frac{t}{2}\right)\theta_2\right] \\ l^*\omega_2 = d\rho = |\nabla\rho\,|\theta_2 \\ l^*\omega_3 = \rho\,[\,dt + \theta_{34}\,]\,. \end{cases} \tag{3.19}$$

In the following we are going to compute $l^*\omega_{j4}, \quad j = 1, 2, 3.$ Note that

$$\begin{aligned} l^*\omega_{14} &= \langle dJ\upsilon_{\nu_t}, \upsilon_{\nu_t}\rangle = \left\langle de^{it/2}J\upsilon_\nu, e^{it/2}\upsilon_\nu\right\rangle \\ &= \left\langle e^{it/2}\left[dJ\upsilon_\nu - \upsilon_\nu\, dt/2\right], e^{it/2}\upsilon_\nu\right\rangle \\ &= \theta_{21} - dt/2. \end{aligned} \tag{3.20}$$

$$\begin{aligned} l^*\omega_{24} &= \langle d\nu_t, \upsilon_{\nu_t}\rangle = \left\langle de^{it}\nu, e^{it/2}\upsilon_\nu\right\rangle \\ &= \left\langle e^{it}[\, d\nu + J\nu\ dt], e^{it/2}\upsilon_\nu\right\rangle \\ &= \left\langle e^{it/2}d\nu, \upsilon_\nu\right\rangle \\ &= \cos(t/2)\ \theta_{31} + \sin(t/2)\ \theta_{41} \\ &= -\rho^{-1}\left[\, \cos(t/2)\ \theta_1 + \sin(t/2)\ \theta_2\, \right]. \end{aligned} \tag{3.21}$$

Similarly we obtain

$$l^*\omega_{34} = -\rho^{-1}[-\sin(t/2)\ \theta_1 + \cos(t/2)\ \theta_2]. \tag{3.22}$$

It follows from (3.19), (3.21) and (3.22) that

$$\begin{cases} 2\rho\omega_{34} = -\omega_1 \\ 2\rho\omega_{24} = \cot(t/2)\ \omega_1 - 2\left[|\nabla\rho|\sin(t/2)\right]^{-1}\omega_2. \end{cases} \tag{3.23}$$

To express ω_{14} in terms of the coframe field $\omega_1, \omega_2, \omega_3$ we first observe that

$$0 = l^*d\omega_4 = \sum_j l^*\omega_j \wedge l^*\omega_{j4} = \Theta \wedge [\sin(t/2)\theta_1 - \cos(t/2)\theta_2],$$

where $\Theta = \theta_{34} - 2\theta_{12} - |\nabla(\ln\rho)|\ \theta_1$. Since this is true for all $0 < t < 2\pi$, it follows that

$$2\theta_{12} - \theta_{34} + \rho^{-1}|\nabla\rho|\theta_1 = 0. \tag{3.24}$$

This allow us to rewrite equation (3.20) as

$$2\rho l^*\omega_{14} = |\nabla\rho|\theta_1 - \rho(\theta_{34} + dt). \tag{3.25}$$

Using equations (3.19) and (3.25) we obtain

$$2\rho\sin(t/2)\ \omega_{14} = -2^{-1}|\nabla\rho|\omega_1 + \cos(t/2)\ \omega_2 - \sin(t/2)\ \omega_3. \tag{3.26}$$

At the given point $X^* = X + \rho(X)\xi_X \in F$ we may write the unit normal vector ξ_X as $\xi_X = e^{it}\nu$ for some $t \in (0, 2\pi)$. Since $\upsilon_\xi = e^{it/2}\upsilon_\nu$, it follows that

$$\begin{cases} \langle\nabla\rho, \upsilon_\xi\rangle = \upsilon_\xi.\rho = |\nabla\rho|\sin(t/2) \\ \langle\nabla\rho, J\upsilon_\xi\rangle = J\upsilon_\xi.\rho = |\nabla\rho|\cos(t/2). \end{cases}$$

The second fundamental form II can be obtained from the following expressions.

$$\begin{cases} 2\rho(\upsilon_\xi.\rho)\,\omega_{14} = -2^{-1}|\nabla\rho|^2\omega_1 + (J\upsilon_\xi.\rho)\omega_2 - (\upsilon_\xi.\rho)\omega_3 \\ 2\rho(\upsilon_\xi.\rho)\,\omega_{24} = (J\upsilon_\xi.\rho)\omega_1 - 2\omega_2 \\ 2\rho\,\omega_{34} = -\omega_1. \end{cases}$$

□

4 Proof of Theorem 1.3

Proof. In the proof, we will use the notation introduced in the previous sections. For this, choose we $0 < t < 2\pi$ and let

$$X^* = X + \rho(X)e^{it}\nu$$

be a point in $F_\Sigma = F(\Sigma) - \Sigma^*$. We know from Lemma 4 that at X^*, the second fundamental form II with respect to the orthonormal frame e_A is given by

$$(\upsilon_\xi.\rho^2)II = \begin{bmatrix} -|\nabla\rho|^2/2 & (J\upsilon_\xi).\rho & -\upsilon_\xi.\rho \\ & -2 & 0 \\ & & 0 \end{bmatrix},$$

where $\xi = e^{it}\nu$. The Gauss-Kronecker K of F_Σ is given by the determinant of the symmetric matrix II. Then

$$K = \left(4\rho^3\upsilon_\xi.\rho\right)^{-1}.$$

Since $\upsilon_\xi.\rho = |\nabla\rho|\sin(t/2) > 0$, it follows that F_Σ is a strictly convex hypersurface. To compute the scalar curvature of F_Σ we first notice that

$$\begin{cases} (\upsilon_\xi.\rho^2)\text{trace}II = -(4 + |\nabla\rho|^2)/2 \\ (\upsilon_\xi.\rho^2)^2\text{trace}II^2 = (4 + |\nabla\rho|^2)^2/4. \end{cases}$$

To complete the proof of Theorem 1.3 we observe that the scalar curvature κ of F is given by

$$\kappa/6 = (\text{trace}II)^2 - \text{trace}II^2 = 0.$$

□

5 The Alexandroff-Fenchel-Jessen Theorem. Proof of Theorem 1.2

Let $k_1, k_2, ..., k_n$ be the principal curvatures of a strictly convex n-dimensioanal hypersurface M of $\mathbb{R}^{n+1}$. As usual let $P_r(M)$ denote the r^h elementary symmetric function of the radii of principal curvatures $1/k_1, ..., 1/k_n$. Note that

$$\binom{n}{r} P_r(M) = \sum_{i_1 < \cdots < i_r} \frac{1}{k_{i_1}} \frac{1}{k_{i_2}} \cdots \frac{1}{k_{i_r}}.$$

For each $1 \leq r \leq n$, we let H_r be r-th mean curvatures of M. We set $H_0 = 1$ and note that for each $0 \leq r < n$,

$$P_{n-r}(M) = \frac{H_r(M)}{H_n(M)} \tag{5.1}$$

Now we recall the following uniqueness theorem of Alexandroff-Fenchel-Jessen.

Theorem 5.1. *Two closed strictly convex hypersurfaces of $\mathbb{R}^{n+1}$ differ by a translation $T : \mathbb{R}^n \to \mathbb{R}^n$ if P_s, $1 \leq s \leq n$ takes the same value at points with the same normal vector.*

Proof. See Chern, [3]. □

In [3], S. S. Chern emphasized that Theorem 5.1 can be extended to hypersurfaces with boundaries. For this, it is necessary that the boundaries differ by a translation and that corresponding points have the same normal vectors. We will refer to the next result as the extended Alexandroff-Fenchel-Jessen Theorem.

Theorem 5.2. *For each $i = 1, 2$, let $M_i \subset \mathbb{R}^{n+1}$ be a strictly convex compact hypersurface with boundary ∂M_i. If*

a) $P_s(M_1) = P_s(M_2)$, for some $1 \leq s \leq n$.

b) $T(\partial M_1) = \partial M_2$ for some translation $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$.

c) the boundaries have the same orientations and the same normal vectors at corresponding points.

Then $T(M_1) = M_2$.

We note that Theorem 1.2 is a consequence of the slightly more general result.

Theorem 5.3. *For each* $i = 1, 2$*, let* $M_i \subset \mathbb{R}^{n+1}$ *be a strictly convex compact hypersurface with boundary* ∂M_i*. If*

a) $H_r(M_1)/H_n(M_1) = H_r(M_2)/H_n(M_2)$*, for some* $0 \le r < n$

b) $T(\partial M_1) = \partial M_2$ *for some translation* $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$*.*

c) the boundaries have the same orientations and the same normal vectors at corresponding points.

Then $T(M_1) = M_2$*.*

Proof. We know from equation 5.1 that $P_{n-r}(M_1) = P_{n-r}(M_2)$. Since $1 \le n - r \le n$, the results follows from the extended Alexandroff-Fenchel-Jessen Theorem. □

6 Final comments

In general a solution of equation

$$\epsilon_r(f) =: \sum_{i_1 < \cdots < i_r} D_{i_1 \cdots i_n}(f) = 0, \quad \det f_{**} \neq 0. \tag{6.1}$$

on a bounded domain with smooth boundary is completely determined by the values of f and ∇f on the boundary. This is the content of the following theorem.

Theorem 6.1. *Let* $f, g : \Omega \to \mathbb{R}$ *be solutions of equation (6.1) in a bounded domain* $\Omega \subset \mathbb{R}^{n+1}$ *with smooth boundary* $\partial\Omega$*. Suppose in addition that in the boundary* $|f - g| + |\nabla(f - g)| = 0$*. Then* $f \equiv g$*.*

Proof. Let ν_f and ν_g be the vector fields given by

$$\begin{cases} W\nu_f = (\nabla f, -1) \\ W\nu_g = (\nabla g, -1) \end{cases} \tag{6.2}$$

They are the unit normals to Γ_f and Γ_g respectively and coincide on their common boundary. Obviously Γ_f and Γ_g induce the same orientation on their common boundary. By assumption Γ_f and Γ_g are strictly convex hypersurfaces with $H_r(\Gamma_f) = H_r(\Gamma_g) = 0$. Using Theorem 1.2 we see that $\Gamma_f = \Gamma_g$ and $f = g$.

$\square$

In this paper we exhibit a special family $\Im$ of 3−dimensional hypersurfaces of $\mathbb{R}^4$ with $H_2 \equiv 0$. The family

$$\Im = (F_\Sigma)_{\Sigma\in\Lambda}$$

was indexed by the set Λ of non-singular analytic curves in $\mathbb{C}^2$. From each $F_\Sigma \in \Im$ we obtain a chain of scalar flat hypersurfaces

$$M_n(\Sigma) =: F_\Sigma \times \mathbb{R}^n$$

of $\mathbb{R}^{n+4}$. With this notation we have the following result.

Theorem 6.2. *Let $M \in \Im_n = \{M_n(\Sigma) : F_\Sigma \in \Im\}$. Then $H_2(M) = 0$ and M is a scalar flat hypersurfaces of $\mathbb{R}^{n+4}$.*

Question 1. For each $k = 1, \cdots, n$ let $X_k : \Sigma_k \to \mathbb{C}^2$ be a non-singular analytic curve with focal locus F_k. What is the geometry of the product $F = F_1 \times \cdots \times F_n$ as a codimension-n submanifold of $\mathbb{C}^{2n}$

Question 2. What can we say about the structure of the focal locus of a complex curve $X : \Sigma \to \mathbb{C}^n$.

These and other questions will be addressed in another occasion.

References

[1] S. C. de Almeida, On the structure of the focal locus of a complex curve, Matemática Contemporânea, **4**, (1993) 5-20.

[2] L. Cafarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for non-linear second order elliptic equations, III: Functions of the eigenvalues of the Hessian, Acta Math. **155**, (1985) 261-301.

[3] S.S. Chern, Integral Formulas for Hypersurfaces in Euclidean Space and Their Applications to Uniqueness Theorems, Journal of Mathematics and Mechanics, **8** (1959), 269-277.

[4] P. do Carmo and M. Dajczer, Rotation hypersurfaces in spaces of constant curvature, Trans. Amer. Math. Soc., **277** (1983), 685-709.

[5] R. Geroch, General Relativity. Proceedings of Symposia in Pure Mathematics, Diff. Geometry, AMS, **XXVII**, (1975).

[6] B. Guan and P.Guan, Convex hypersurfaces of prescribed curvature, Ann Math, 156 (2002), 655-673.

[7] P.Guan, Xi-Nan Ma, N, Trudinger and X. Zhu, A Form of Alexandrov-Fenchel Inequality, Pure and Applied Mathematics Quartely, 6 (2010), 999-1012.

[8] J. Hounie and M. L. S. Leite, The maximum principle for hypersurfaces with $H_r = 0$. Journal of Differential Geometry, **41**, (1995), 247-258.

[9] J. Hounie and M. L. S. Leite, Uniqueness and nonexistence theorems for hypersurfaces with $H_r = 0$. Annals of Global Analysis and Geometry, **17**, (1999) 397-407.

[10] J. Hounie and M. L. S. Leite, Two-ended hypersurfaces with zero scalar curvature. Indiana University Mathematics Journal. **48**, (1999), 867-892.

[11] H. B. Lawson, Jr., Lectures on Minimal Submanifolds, IMPA, Rio de Janeiro, (1973). Reprinted by Publish or Perish, (1980).

[12] W. Y. Hsiang, On rotational W-hypersurfacesbin spaces of constant curvature and the generalizaed laws of sine and cosine, Bull. Inst. Math. Acad. Sinica **11**, (1983), 349-373.

[13] M. L. S. Leite, Geroch's conjecture for a hypersurface of $\mathbb{R}^4$. Bulletin of the Brazilian Mathematical Society, **8**, (1977), 121-126.

[14] M. L. S. Leite, Rotational hypersurfaces with constant scalar curvature. Manuscripta Mathematica, **67**, (1990), 285-304.

[15] M. L. S. Leite, An example of a triply periodic complete embedded scalar-flat hypersurface of $\mathbb{R}^4$. Anais da Academia Brasileira de Ciências, **63**, (1991), 383-385.

[16] H. Mori, Rotational hypersurfaces in $\mathbb{S}^n$ and $\mathbb{H}^n$ with constant scalar curvature, Yokohama Mathematical Journal, **39**, (1992), 151-162.

[17] O. Palmas, Complete rotation hypersurfaces with H_k constant in space forms, Bol. Soc. Bras. Mat., **30**, (1999), 139-161.

[18] W. Sheng, N. Trudinger and X.J. Wang, Convex Hypersurfaces of Prescribed Weingarten Curvatures, Comunications in Analysis and Geometry, **12**, (2004), 213-232.

[19] N. Trudinger, The Dirichlet problem for the prescribed curvature equations, Arch. Rational Mech. Anal., **111**, (1990), 153-179.

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