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On the Ricci curvature equation and the Einstein equation for diagonal tensors

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Abstract

We consider the pseudo-euclidean space (R^n, g), with $n \geq 3$. We provide necessary and sufficient conditions for a diagonal tensor to admit a metric $\bar{g}$, conformal to g, that solves the Ricci tensor equation or the Einstein equation. Examples of complete metrics are included.

Introduction

We consider the following two general problems. Given a symmetric tensor T, of order two, defined on a manifold M^n, $n \geq 3$, does there exist a Riemannian metric g such that $Ric\, g = T$? Find necessary and sufficient conditions on a symmetric tensor T, so that one can find a metric g satisfying $Ric\, g - \frac{K}{2} g = T$, where K is the scalar curvature of g. Both problems correspond to solving nonlinear differential equations. The first one we call the Ricci tensor equation and the second one the Einstein equation.

DeTurck [D1] showed that, when T is nonsingular, a local solution of the Ricci equation always exists. The singular case, with constant rank and additional conditions, was considered by DeTurck-Goldschmidt [DG]. Rotationally symmetric nonsingular tensors were considered by Cao-DeTurck [CD]. Other results were obtained by DeTurck [D2], DeTurck-Koiso [DK], Lohkamp [L] and Hamilton [H].

DeTurck [D3] also considered the Cauchy problem for nonsingular tensors for the Einstein field equation, i.e. $n = 4$. For other results, when T represents

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several physical situations, we reffer the reader to [SKMHH] and its references
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In our previous papers, [P, PT1-PT6], we investigated both the Ricci equation and the Einstein equation, for the following special classes of tensors T and metrics conformal to the pseudoeuclidean metric g. In [PT1, PT2], we considered symmetric tensors of type $T = \sum \varepsilon_i c_{ij} dx_i dx_j$ where $\varepsilon_i = \pm 1$ and c_{ij} are real constants. In [PT3, PT4], we studied tensors $T = fg$ where f is a real function. Diagonal tensors depending on one variable were considered in [PT5] and tensors $T = \sum_{i,j} f_{ij} dx_i dx_j$ whose nondiagonal terms $f_(x_i, x_j)$ depend on x_i, x_j were investigated in [PT6].

In this paper, we consider diagonal tensors T on a pseudo-euclidean space (R^n, g), $n \geq 3$, and we provide necessary and sufficient conditions for the existence of a metric conformal to g, whose Ricci tensor is a given tensor T. A similar question is considered for the Einstein equation. The theory is also extended to locally conformally flat manifolds.

More precisely, we consider the pseudo-euclidean (R^n, g), with $n \geq 3$, coordinates $x = (x_1, .., x_n)$ and $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$, where at least one ϵ_i is positive. We consider diagonal tensors of the form $T = \sum\limits_i \epsilon_i f_i(x) dx_i^2$, where $f_i(x)$ is a differentiable function. For such a tensor, we want to find metrics $\bar{g} = g/\varphi^2$, that solve the Ricci equation or the Einstein equation.

Our main results in this paper assume that not all the functions f_i to be equal and not all to be constant, since we studied the case when all functions f_i are constant in [PT1] and [PT2] and we investigated the case when all functions f_i are equal in [PT4]. For the sake of completeness we include these results in Section 1.

Our first theorem (Theorem 1.1) gives a characterization of such tensors when the functions f_i depend on r variables where $1 < r < n$. Theorems 1.2 and 1.3 give necessary and sufficient conditions, in terms of ordinary differential equations, for the existence of conformal metrics for the Ricci and Einstein equations. As a consequence of Theorem 1.2, we show that for certain functions $\bar{K}$, depending on functions of one variable, $U_j(x_j)$, there exist metrics $\bar{g}$, conformal to the pseudo- euclidean metric g, whose scalar curvature is $\bar{K}$. This result is related to the prescribed scalar curvature problem: Given a differentiable function $\bar{K}$, on a Riemannian manifold (M, g), is there a metric $\bar{g}$ conformal to g whose scalar curvature is $\bar{K}$? This problem has been studied by many authors. In particular, when $\bar{K}$ is constant, it is known as the Yamabe problem.

By applying the theory, we exhibit examples of complete metrics on R^n, on the n-dimensional torus T^n, or on cylinders $T^k \times R^{n-k}$, that solve the Ricci equation or the Eisntein equation.

1 Main results

We will now state our main results. The proofs will be given in the next section. We will consider diagonal tensors $T = \sum_{i=1}^{n} \varepsilon_i f_i(x) dx_i^2$ on a pseudo-euclidean space, (R^n, g), $n \geq 3$, with coordinates $x = (x_1, ..., x_n)$, and metric $g_{ij} = \delta_{ij}\epsilon_i$, where $\epsilon_i = \pm 1$. We wil assume that not all f_i are constant and not all are equal. Our results will complete the study of solving the Ricci and Einstein equations, in the conformal class, for diagonal tensors, in the pseudo-euclidean space. For the sake of completeness, we will include in this section the corresponding results for the case when all f_i are constants and when they are all equal. These were solved in our previous papers [PT1], [PT2] and [PT4]. We will denote by $\varphi_{,ij}$ and $f_{i,k}$ the second order derivative of φ with respect to $x_i x_j$ and the derivative of f_i with respect to x_k, respectively.

Our first result considers tensors whose diagonal elements depend on $r < n$ variables.

Theorem 1.1 *Let (R^n, g), $n \geq 3$, be the pseudo-euclidean space, with coordinates $x_1, ..., x_n$, and metric $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Let $T = \sum_{i=1}^{n} \varepsilon_i f_i(\hat{x}) dx_i^2$, be a diagonal tensor such that the functions f_i depend on $\hat{x} = (x_1, ..., x_r)$ where $1 < r < n$. Assume not all f_i to be constant and not all to be equal and let $F_i = f_i - f_n$ $\forall, i < n$. Let $W \subset R^{n-1}$ be an open set such that $I = \{i < n;\, F_i(\hat{x}) \neq 0, \forall \hat{x} \in W\}$ is non empty. Then there exists a conformal metric $\bar{g} = \frac{1}{\varphi^2} g$ such that Ric $\bar{g} = T$ or Ric $\bar{g} - \frac{\bar{K}}{2}\bar{g} = T$ if, and only if, for all distinct indices $i, j, k \in I$,*

$$\left(\ln \frac{F_i}{F_k}\right)_{,j} = 0, \qquad \left(\ln \frac{F_i}{F_j}\right)_{,ij} = 0, \tag{1.1}$$

and for all $r \notin I$, $\varphi_{,rr} = 0$.

Our next two results give a characterization of our problems in terms of systems of ordinary differential equations.

Theorem 1.2 *Let* (R^n, g), $n \geq 3$, *be a pseudo-euclidean space, with coordinates* $x = (x_1, ..., x_n)$, *and* $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. *Consider a diagonal tensor* $T = \sum_{i=1}^{n} \varepsilon_i f_i(x) dx_i^2$. *Assume not all the functions* f_i *to be equal and not all to be constant. Then there exists a metric* $\bar{g} = \frac{1}{\varphi^2} g$ *such that Ric* $\bar{g} = T$ *if, and only if, there exist functions* $U_j(x_j)$, $1 \leq j \leq n$ *which satisfy the system of differential equations*

$$U_i'' = \frac{\epsilon_i}{n-2} \left(f_i - \frac{\sum_{s=1}^{n} f_s}{2(n-1)} \right) \sum_{s=1}^{n} U_s + \frac{\epsilon_i \sum_{s=1}^{n} \epsilon_s (U_s')^2}{2 \sum_{s=1}^{n} U_s} \tag{1.2}$$

and $\varphi = \sum_{s=1}^{n} U_s(x_s)$. *In particular, if* $f_i = f_j$ *for* $i \neq j$ *then* U_i *and* U_j *are quadratic functions in* x_i *and* x_j *respectively. Moreover, if all functions* f_i *do not depend on a variable* x_s, *then* U_s *is constant.*

Theorem 1.3 *Let* (R^n, g), $n \geq 3$, *be a pseudo-euclidean space, with coordinates* $x = (x_1, ..., x_n)$, *and* $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. *Consider a diagonal tensor* $T = \sum_{i=1}^{n} \varepsilon_i f_i(x) dx_i^2$. *Assume not all the functions* f_i *to be equal and not all to be constant. Then there exists a metric* $\bar{g} = \frac{1}{\varphi^2} g$ *such that Ric* $\bar{g} - \frac{\bar{K}}{2}\bar{g} = T$ *if, and only if, there exist functions* $U_j(x_j)$, $1 \leq j \leq n$ *which satisfy the system of differential equations*

$$U_i'' = \frac{\epsilon_i}{n-2} \left(f_i - \frac{\sum_{s=1}^{n} f_s}{n-1} \right) \sum_{s=1}^{n} U_s + \frac{\epsilon_i \sum_{s=1}^{n} \epsilon_s (U_s')^2}{2 \sum_{s=1}^{n} U_s} \tag{1.3}$$

and $\varphi = \sum_{s=1}^{n} U_s(x_s)$. *In particular, if* $f_i = f_j$ *for* $i \neq j$ *then* U_i *and* U_j *are quadratic functions in* x_i *and* x_j *respectively. Moreover, if all functions* f_i *do not depend on a variable* x_s, *then* U_s *is constant.*

We observe that a particular case of Theorems 1.2 and 1.3 was obtained in [PT5], when the the functions f_i of the tensor T depend on one variable.

Corollary 1.4 *If (R^n, g) is the Euclidean space and $0 < |\varphi(x)| \leq C$ for some constant C, then the metrics given by Theorems 1.2 and 1.3 are complete on R^n.*

Before going on with our results, for the sake of completness, we will state the theorems analogous to Theorems 1.2 and 1.3 in the cases when the functions f_i of the tensor T are either all equal or they are all constants. The next theorem considers the case when the functions f_i of the tensor T are all equal.

Theorem [PT4] *Let (R^n, g), $n \geq 3$, be a pseudo-euclidean space, with coordinates $x = (x_1, ..., x_n)$, and $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Then there exists $\bar{g} = \frac{1}{\varphi^2} g$ such that Ric $\bar{g} = fg$, (resp. Ric $\bar{g} - \frac{\bar{K}}{2}\bar{g} = fg$) if, and only if,*

$$\varphi(x) = \sum_{i=1}^{n} (\epsilon_i a x_i^2 + b_i x_i) + c$$

$$f(x) = \frac{-(n-1)}{\varphi^2}\lambda, \qquad (resp.\ f(x) = \frac{(n-1)(n-2)}{2\varphi^2}\lambda,),$$

where a, b_i, c are real numbers and $\lambda = \sum_i \epsilon_i b_i^2 - 4ac$. Any such metric $\bar{g}$ is unique up to homothety. Whenever g is the euclidean metric then:

a) If $\lambda < 0$ then $\bar{g}$ is globally defined on R^n and T is positive (resp. negative) definite.

b) If $\lambda \geq 0$ then, excluding the homothety, the set of singularity points of $\bar{g}$ consists of

b.1) a point if $\lambda = 0$;

b.2) a hyperplane if $\lambda > 0$ and $a = 0$;

b.3) an $(n-1)$-dimensional sphere if $\lambda > 0$ and $a \neq 0$.

The next theorems consider the case when the functions f_i of the non zero tensor T are all constant.

Theorem [PT1] *Let (R^n, g) be a pseudo-Euclidean space and let $T = \sum_{i=1}^n \varepsilon_i c_i dx_i^2$ be a non zero diagonal tensor. Then there exists $\bar{g} = g/\varphi^2$ such that Ric $\bar{g} = T$ if, and only if, there exits k, $1 \leq k \leq n$ and $b \in R$, such that $c_k = 0$, $b\varepsilon_k < 0$ and $T_k = b\sum_{i \neq k} \varepsilon_i dx_i^2$. In this case, up to homothety, $\varphi = exp(\pm\sqrt{\dfrac{-b\varepsilon_k}{n-2}})\, x_k$.*

Theorem [PT2] *If $T = \sum_{i=1}^n \varepsilon_i c_i dx_i^2$ is a non zero diagonal tensor, then there exists a solution $\bar{g}$ such that Ric $\bar{g} - \bar{K}\bar{g}/2 = 0$ if, and only if, there exits k, $1 \leq k \leq n$ and $b \in R$, such that $b\varepsilon_k > 0$ such that*

$$T = \begin{cases} b\varepsilon_k dx_k^2 & \text{if } n = 3, \\ b\displaystyle\sum_{i \neq k, i=1}^{n} \varepsilon_i dx_i^2 + \frac{n-1}{n-3} b\varepsilon_k dx_k^2 & \text{if } n \geq 4. \end{cases}$$

In this case, up to homothety,

$$\varphi = \begin{cases} \exp(\pm\sqrt{b\varepsilon_k}\, x_k) & \text{if } n = 3, \\ \exp\left(\pm\sqrt{\dfrac{2b\varepsilon_k}{(n-2)(n-3)}}\, x_k\right) & \text{if } n \geq 4. \end{cases}$$

The next theorem considers the case when the tensor $T = 0$.

Theorem [PT1] [PT2] *Let (R^n, g) be a pseudo-Euclidean space. Then there exists $\bar{g} = g/\varphi^2$ such that Ric $\bar{g} = 0$ or Ric $\bar{g} - \bar{K}\bar{g}/2 = 0$ if, and only if,*

$$\varphi = \sum_{j=1}^{n} (a\varepsilon_j x_j^2 + b_j x_j) + c, \qquad \textit{where} \qquad 4ac - \sum_j \varepsilon_j b_j^2 = 0$$

and a, c, b_j are real constants. In both cases, $\bar{K} \equiv 0$, i.e. Ric $\bar{g} \equiv 0$.

We will now state corollaries of Theorem 1.2 obtained by considering $u = \varphi^{-(n-2)/2}$ and the expression of the scalar curvature obtained from the Ricci tensor T, These corollaries are related to the prescribed scalar curvature problem, as one can see in Corollary 1.6.

Corollary 1.5 *Let (R^n, g) be a pseudo-euclidean space, $n \geq 3$, with coordinates $x = (x_1, ..., x_n)$, $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Let $\bar{K} : R^n \to R$ be given by*

$$\bar{K} = (n-1)\left\{ 2(\sum_{s=1}^{n} \epsilon_s U_s) \sum_{s=1}^{n} U_s'' - n\sum_{s=1}^{n} \epsilon_s (U_s')^2 \right\}. \tag{1.4}$$

where $U_j(x_j)$, $1 \leq j \leq n$, *are arbitrary nonconstant differentiable functions. Then the differential equation*

$$\frac{4(n-1)}{n-2}\Delta_g u + \bar{K}(x)u^{\frac{n+2}{n-2}} = 0 \tag{1.5}$$

where Δ_g *denotes the laplacian in the metric* g, *has a solution, globally defined on* R^n, *given by*

$$u = \left(\sum_{s=1}^{n} \frac{\epsilon_s U_s}{n-2}\right)^{-\frac{n-2}{2}}. \tag{1.6}$$

The geometric interpretation of the above results is the following:

Corollary 1.6 *Let* (R^n, g) *be a pseudo-euclidean space,* $n \geq 3$ *and* $\bar{K}$ *a function given by (1.4). Then there exists a metric* $\bar{g} = u^{\frac{4}{n-2}} g$, *where* u *is given by (1.6), whose scalar curvature is* $\bar{K}$. *In particular, if* (R^n, g) *is the euclidean space and* u *is a bounded function then* $\bar{g}$ *is a complete metric.*

Examples 1.7 As a direct consequence of Theorems 1.2 and 1.3 and Corollary 1.4, we get the following examples, where we are considering (R^n, g), $n \geq 3$, the pseudo-euclidean space with coordinates $(x_1, ..., x_n)$ such that $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$.

a) Consider for each $j = 1, ..., n$, the function $U_j = \exp(-x_j^{2m_j})$, where m_j is a positive integer and the tensor T determined as in Theorem 1.2 by (1.2). We observe that although this tensor may have singular points (depending on the integers m_j), there exists $\bar{g} = \frac{1}{\varphi^2} g$ such that $Ric\, \bar{g} = T$, globally defined on R^n with $\varphi = \exp(-\sum_j x_j^{2m_j})$. Moreover, it follows from Corollary 1.4, that in the euclidean case, the metric $\bar{g}$, is a complete metric on R^n.

b) Consider any periodic nonconstant function $U_j(x_j)$ for each $j = 1, ..., n$. Then the symmetric tensor $T = \sum_{i=1}^{n} f_i(x_1, ..., x_n) dx_i^2$, defined as in Theorem 1.2, admits a metric $\bar{g}$, on an n-dimensional torus, T^n, conformal to the pseudo-euclidean metric, whose Ricci tensor is T. Observe that in the Euclidean case ($\epsilon_k = 1, \forall k$), $\bar{g}$ is a complete metric on T^n. If we consider k periodic functions U_j, we get metrics defined on $T^k \times R^{n-k}$, conformal to the pseudo-euclidean metric. In the euclidean case, if moreover φ is a bounded function, then $\bar{g}$ is a complete metrics on $T^k \times R^{n-k}$.

c) As a consequence of Theorem 1.3, we observe that periodic functions $U_j(x_j)$, for each $j = 1, ..., n$, determine a tensor T which admits a solution $\bar{g}$, conformal to g, for the Einstein equation, defined on T^n. If we consider k periodic functions U_j, $k < n$, we get solutions for the Einstein equation on $T^k \times R^{n-k}$. In the Euclidean case, if moreover φ is a bounded function, then $\bar{g}$ is a complete metric.

We now consider a Riemannian manifold locally conformally flat (M^n, g). It is easy to see that the following results hold.

Corollary 1.8 *Let (M^n, g), $n \geq 3$ be Riemannian manifold, locally conformally flat. Let V be an open subset of M with coordinates $x = (x_1, ..., x_n)$ such that $g_{ij} = \delta_{ij}/F^2$. Consider a diagonal symmetric tensor $T = \sum_{i=1}^{n} f_i(x)dx_i^2$. Assume not all functions f_i to be equal and not all to be constant. Then there exists $\bar{g} = \frac{1}{\psi^2}g$ such that Ric $\bar{g} = T$ if, and only if, there exist $U_j(x_j)$, $1 \leq j \leq n$ differentiable functions such that, U_j and φ are given as in Theorem 1.2 and $\psi = \frac{\varphi}{F}$.*

The following result provides the analogue theorem for the Einstein equation.

Corollary 1.9 *Let (M^n, g), $n \geq 3$, be Riemannian manifold, locally conformally flat. Let V be an open subset of M with coordinates $x = (x_1, ..., x_n)$ such that $g_{ij} = \delta_{ij}/F^2$. Consider a diagonal symmetric tensor $T = \sum_{i=1}^{n} f_i(x)dx_i^2$. Assume not all functions f_i to be equal and not all to be constant. Then there exists a metric $\bar{g} = \frac{1}{\psi^2}g$ such that Ric $\bar{g} - \frac{\bar{K}}{2}\bar{g} = T$ if, and only if, there exist $U_j(x_j)$, $1 \leq j \leq n$ differentiable functions such that, U_j and φ are given as in Theorem 1.3 and $\psi = \frac{\varphi}{F}$.*

We observe that there are similar results for manifolds that are locally conformal to the pseudo-euclidean space.

2 Proof of the main results

Before proving our results, we observe that if (R^n, g) is a pseudo-euclidean space and $\bar{g} = g/\varphi^2$ is a conformal metric, then the scalar curvature of $\bar{g}$ is given by

$$\bar{K} = (n-1)\left(2\varphi\Delta_g\varphi - n|\nabla_g\varphi|^2\right). \tag{2.1}$$

Moreover, studying the Ricci and Einstein equations, in the conformal class, when $T = \sum_{i=1}^{n} \epsilon_i f_i(x) dx_i^2$ is equivalent to studying repectively the following systems of equations:

$$\begin{cases} \epsilon_i f_i = \frac{1}{\varphi^2}\{(n-2)\varphi\varphi_{,ii} + (\varphi\triangle_g\varphi - (n-1)|\nabla_g\varphi|^2)\varepsilon_i\} \qquad \forall\ i: 1, ..., n, \\ \varphi_{,ij} = 0 \quad \forall i \neq j, \end{cases} \tag{2.2}$$

$$\begin{cases} \epsilon_i f_i = \frac{1}{\varphi^2}\{(n-2)\varphi\varphi_{,ii} + (-(n-2)\varphi\triangle_g\varphi + \frac{(n-1)(n-2)}{2}|\nabla_g\varphi|^2)\varepsilon_i\} \\ \qquad\qquad \forall \quad i: 1, ..., n. \\ \varphi_{,ij} = 0 \quad \forall i \neq j. \end{cases} \tag{2.3}$$

where $\triangle_g$ and ∇_g denote the laplacian and the gradient in the pseudo-euclidean metric g. It follows from the second and first equations of (2.2) (resp. (2.3)) that $\varphi = \sum_{i=1}^{n} \varphi_i(x_i)$ and

$$\epsilon_i\varphi_i'' - \epsilon_j\varphi_j'' = \frac{(f_i - f_j)}{(n-2)}\varphi, \qquad \forall i \neq j. \tag{2.4}$$

Proposition 1.10 *Let $\varphi(x_1, ..., x_n)$ be a solution of (2.2) or (2.3), where $f_i(\hat{x})$ are functions that depend on $\hat{x} = (x_1, ..., x_r)$ and $r < n$. Assume not all f_i to be constant and not all to be equal. Then $\varphi_{,s} = 0$, $\forall s > r$.*

Proof: If φ is a solution of (2.2) or (2.3), then $\varphi = \sum_{i=1}^{n} \varphi_i(x_i)$ and (2.4) holds for all $i \neq j$. Now we fix s, such that $r < s \leq n$ and consider (2.4) for i, j, s distinct. Taking the derivative with respect to x_s we have

$$(f_i - f_j)\varphi_{,s} = 0 \quad \forall \quad i \neq j \text{ distinct from } s.$$

Assume $\varphi_{,s} \neq 0$ in an open subset $W \subset R^n$. Then $f_i = f_j \quad \forall \quad i \neq j$, distinct from s. It follows from (2.4) that $\epsilon_i\varphi_i'' = \epsilon_j\varphi_j'' \quad \forall i \neq j$, distinct from s in W. Hence, $\varphi_i'' = 2c_i$ and $\varphi_j'' = 2c_j$ in W where $\epsilon_i c_i = \epsilon_j c_j$.

It follows from (2.4) that

$$\epsilon_s \varphi_s^{''} - 2\epsilon_i c_i = \frac{(f_s - f_i)}{(n-2)} \varphi \quad \forall \qquad i \neq s \tag{2.5}$$

Taking the derivative of (2.5) with respect to x_j with $j \leq r$, we have

$$(f_s - f_i)_{,j} \varphi + (f_s - f_i) \varphi_{,j} = 0 \quad \forall \qquad i \neq s,\, j \leq r. \tag{2.6}$$

If there exists $i_0 \neq s$ such that $f_s - f_{i_0}$ is not a constant in $V \subset W$, then there exists $j_0 \leq r$ such that

$$\varphi = \frac{(f_s - f_{i_0})}{(f_s - f_{i_0})_{,j_0}} \varphi_{j_0}^{'}$$

in V. Taking the derivative with respect to x_s we get $\varphi_{,s} = 0$, which is a contradiction.

Therefore, $\forall i \neq s$, we have $f_s - f_i = c_i$, where $c_i \in R$ and it follows from (2.6), that $c_i \varphi_j^{'} = 0 \quad \forall \quad j \leq r,\, i \neq s$. Since not all functions f_i are equal, there exists i_0 such that $c_{i_0} \neq 0$. Hence $\varphi_j^{'} = 0, \forall j \leq r$ in W, i.e. φ depends on $x_{r+1}, ..., x_n$. It follows from (2.2) or (2.3) that f_i depend on these variables. However, by hypothesis, f_i depend on $\hat{x}$. Therefore, we conclude that all functions f_i are constant, which is a contradiction on the hypothesis of the proposition.

We conclude that $\varphi_{,s} = 0$, for all $s > r$.

$\square$.

Proof of Theorem 1.1:
Suppose $\bar{g} = g/\varphi$ is a solution of Ric $\bar{g} = T$ or Ric $\bar{g} - \frac{\bar{K}}{2}\bar{g} = T$. Then, φ satisfies (2.2) (resp. (2.3)) and we are in the conditions of Proposition 1.10. Hence $\varphi_{,s} = 0$ for all $s > r$. In particular, $\varphi_{,n} = 0$. It follows from (2.4) that

$$(n-2)\epsilon_i \varphi_i^{''} = (f_i - f_n)\varphi, \quad \forall i < n. \tag{2.7}$$

Taking the derivative with respect to x_k with $k < n$ and $k \neq i$, we have

$$(f_i - f_n)_{,k} \varphi + (f_i - f_n) \varphi_{,k} = 0, \quad 1 \leq i \neq k < n. \tag{2.8}$$

Considering $F_i = f_i - f_n$, if $i \in I$ it follows from (2.8) that the first equality of (1.1) holds for all $i, j \in I$ and $k < n$ distinct from i and j. Moreover, it follows from the commutativity of the second derivative of $\ln \varphi$ that,

$(\ln F_i)_{,ij} = (\ln F_j)_{,ji}$ for all $i \neq j \in I$, which proves the second equality of (1.1).

If $\ell \notin I$, then $F_\ell \equiv 0$ and it follows from (2.7) that $\varphi_{,\ell\ell} = 0$.

Conversely, if (1.1) holds. Then, $\forall i, j \in I$ we have that $\dfrac{F_i}{F_j}$ depends only on x_i and x_j and $\left(\ln \dfrac{F_i}{F_j}\right)_{,ij} = 0$. Hence, $\dfrac{F_j}{F_i}$ is a product of functions of separated variables x_i and x_j. Therefore, there exist differentiable fuctions $U_i(x_i)$ and $U_j(x_j)$ such that $\dfrac{F_j}{F_i} = \dfrac{U_j''(x_j)}{U_i''(x_i)}$. Similarly, for $k, i \in I$, we have $\dfrac{F_k}{F_i} = \dfrac{\tilde{U}_k''(x_k)}{\tilde{U}_i''(x_i)}$. It follows that

$$\frac{F_k}{F_j} = \frac{\tilde{U}_k''(x_k)U_i''(x_i)}{\tilde{U}_j''(x_j)\tilde{U}_i''(x_i)} \tag{2.9}$$

Taking the derivative, with respect to x_i, of the logarithm of (2.9), it follows that $\left(\frac{\tilde{U}_j''(x_j)}{U_i''(x_i)}\right)_i = 0$. Hence, $\tilde{U}_i''(x_i)$ is a multiple of $U_i''(x_i)$. Therefore, for each $i, j \in I$, we have

$$\frac{F_i}{F_j} = C_{ij}\frac{U_j''(x_j)}{U_i''(x_i)}$$

where $C_{ij} \neq 0$ is a real constant.

We conclude that, for each $i \in I$ we have a differentiable function $U_i(x_i)$, and for each $\ell \notin I$, since $\varphi_{,\ell\ell} = 0$, there is a linear function $U_\ell(x_\ell)$.

We define

$$\varphi = \sum_{i \in I} U_i(x_i) + \sum_{\ell \notin I} U_\ell(x_\ell). \tag{2.10}$$

Then $\bar{g} = \dfrac{1}{\varphi^2} g$ is a solution of the Ricci equation $\mathrm{Ric}\, \bar{g} = T$ (respectively the Einstein equation $\mathrm{Ric}\, \bar{g} - \dfrac{\bar{K}}{2}\bar{g} = T$) and the functions f_k of the tensor T are obtained in terms of the functions U_i and U_ℓ by the equations (2.2) (resp. (2.3)).

□

Proof of Theorem 1.2:

The metric $\bar{g} = g/\varphi^2$ satisfies the Ricci equation $\mathrm{Ric}\, \bar{g} = T$ if, and only if, φ satisfies (2.2), i.e. there exist $U_j(x_j)$, $1 \leq j \leq n$ differentiable functions such that $\varphi = \sum_{s=1}^{n} U_s(x_s)$ and f_j are given by

$$f_i = \frac{1}{\sum_{s=1}^n U_s}\left(\epsilon_i(n-2)U_i^{''} + \sum_{s=1}^{n}\epsilon_s U_s^{''}\right) - (n-1)\frac{\sum_{s=1}^{n}\epsilon_s (U_s^{'})^2}{\sum_{s=1}^{n}(U_s)^2}.$$

A straightforward computation shows that this system of equations is equivalent to (1.2).

If $f_i = f_j$ for any pair of indices $i \neq j < n$, then the functions U_i and U_j are quadratic functions in x_i and x_j repectively. In fact, this follows immediately from (2.4).

Moreover, if all functions f_i do not depend on a variable x_s, then, by reordering the variables if necessary, it follows from Proposition 1.10, that φ does not depend on xs and hence U_s is constant.

□

Proof of Theorem 1.3:

The metric $\bar{g} = g/\varphi^2$ satisfies the Ricci equation Ric $\bar{g} - \bar{K}\bar{g}/2 = T$ if, and only if, φ satisfies (2.3), i.e. there exist $U_j(x_j)$, $1 \leq j \leq n$ differentiable functions such that $\varphi = \sum_{s=1}^{n} U_s(x_s)$ and f_j are given by

$$f_i = \frac{n-2}{\sum_{s=1}^n U_s}\left(\epsilon_i U_i^{''} - \sum_{s=1}^{n}\epsilon_s U_s^{''} + (n-1)\frac{\sum_{s=1}^n \epsilon_s (U_s^{'})^2}{2\sum_{s=1}^n (U_s)}\right).$$

A straightforward computation shows that this system of equations is equivalent to (1.3).

If $f_i = f_j$ for any pair of indices $i \neq j$, then the functions U_i and U_j are quadratic functions in x_i and x_j respectively. This follows immediately from (2.4).

Moreover, if all functions f_i do not depend on a variable x_s, then, by reordering the variables if necessary, it follows from Proposition 1.10, that φ does not depend on x_s and hence U_s is constant.

□

Proof of Corollary 1.4:

Consider the Euclidean space (R^n, g), $n \geq 3$ and a metric $\bar{g}$ given by Theorems

1.2 or 1.3. If $0 < |\varphi(x)| \leq C$, then the metric $\bar{g}$ is complete, since there exists a constant $m > 0$, such that for any vector $v \in R^n$, $|v|_{\bar{g}} \geq m|v|$.

□

Proof of Corollary 1.5:

It follows from (2.1), that for the metric $\bar{g}$ of Theorem 1.2 the scalar curvature is given by (1.4). By defining the function $u^{\frac{-2}{n-2}} = \varphi$, we conclude that u is a solution of (1.5).

□

Proof of Corollary 1.6:

This result follows immediately from the previous corollaries, since finding a metric $\bar{g} = u^{\frac{4}{n-2}} g$, with scalar curvature $\bar{K}$ is equivalent to solving equation (1.5).

□

In order to prove Corollaries 1.8 and 1.9, we consider $\psi = \varphi F$ and apply Theorems 1.2 and 1.3.

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