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Some characterizations of Euclidean domains in the steady state and hyperbolic spaces

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To Prof. Antonio Gervásio Colares on occasion of his 80^{th} birthday

Abstract

Under suitable restrictions on the image of the Gauss mapping and on the values of the mean curvature, we extend the technique developed by Colares jointly with the second author in [7], in order to establish characterization results concerning the Euclidean domains of the steady state space $\mathcal{H}^{n+1}$ and of the hyperbolic space $\mathbb{H}^{n+1}$. As applications of such characterizations, we obtain rigidity theorems for the spacelike hyperplanes of $\mathcal{H}^{n+1}$ and for the horospheres of $\mathbb{H}^{n+1}$.

1 Introduction

In the last years, many authors have approached the problem of estimating the height function of a compact hypersurface whose boundary is contained into an umbilical hypersurface of a certain ambient space. The first result in this direction is due by Heinz [11] who proved that a compact graph of positive constant mean curvature H in the 3-dimensional Euclidean space $\mathbb{R}^3$ with boundary on a plane can reach at most height $\frac{1}{H}$ from the plane. Later on, an optimal bound was also obtained for compact graphs and also for compact embedded surfaces with constant mean curvature and boundary on a plane in the 3-dimensional hyperbolic space $\mathbb{H}^3$ by Korevaar, Kusner, Meeks and

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Solomon [13]. Afterwards, Rosenberg [23] extended these previous results for the case of a compact embedded hypersurface with a positive constant r-th mean curvature in $\mathbb{R}^{n+1}$ and in $\mathbb{H}^{n+1}$.

In the Lorentzian setting, López obtained in [16] a sharp estimate for the height of compact spacelike surfaces immersed into the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ with constant mean curvature. For the case of constant higher order mean curvature, by applying the techniques used by Hoffman, de Lira and Rosenberg in [8], the second author obtained in [15] another sharp height estimate for compact spacelike hypersurfaces immersed in $\mathbb{L}^{n+1}$ with a positive constant r-th mean curvature. As an application of such estimate, he studied the nature of the end of a complete spacelike hypersurface in $\mathbb{L}^{n+1}$.

More recently, the second author jointly with Colares [7] obtained height estimates concerning to a compact spacelike hypersurface Σ^n immersed with constant mean curvature H in the half $\mathcal{H}^{n+1}$ of the de Sitter space which models the so-called *steady state space*, when its boundary is contained into some hyperplane of $\mathcal{H}^{n+1}$. Moreover, they applied their estimates to describe the end of a complete spacelike hypersurface and to get theorems of characterization concerning spacelike hyperplanes of $\mathcal{H}^{n+1}$.

In this paper, we extend the technique developed in [7] in order to establish characterization results concerning the *Euclidean domains* of the steady state space $\mathcal{H}^{n+1}$ and of the hyperbolic space $\mathbb{H}^{n+1}$ (that is, domains entirely contained in a spacelike hyperplane of $\mathcal{H}^{n+1}$ and in a horosphere of $\mathbb{H}^{n+1}$, respectively). More precisely, we prove the following (cf. Theorems 3.3 and 4.1, respectively):

Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a compact spacelike hypersurface with H_2 constant and whose boundary $\partial\Sigma$ is contained in a hyperplane $\mathcal{L}_\tau$, for some $\tau > 0$. If the mean curvature H verifies

$$1 \leq H \leq H_2,$$

then Σ^n is an Euclidean domain of $\mathcal{H}^{n+1}$.

Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ be a compact hypersurface, with H_2 constant and whose boundary $\partial\Sigma$ is contained in a horosphere L_ϱ, for some $\varrho > 0$. Suppose that Σ^n lies in L_ϱ^- and that the image of its Gauss mapping $N(\Sigma)$ is contained into

$\mathcal{L}_\tau^+ \subset \mathcal{H}^{n+1}$, *for some* $\tau > 0$. *If the mean curvature* H *verifies*

$$H \geq H_2 \geq \left(\frac{\varrho}{\tau}\right)^2,$$

then Σ^n *is an Euclidean domain of* $\mathbb{H}^{n+1}$.

Here, $H_2 = \frac{2}{n(n-1)} S_2$ denotes the mean value of the second elementary symmetric function S_2 on the eigenvalues of the shape operator of Σ^n. Moreover, L_ϱ^- and $\mathcal{L}_\tau^+$ stand for regions naturally associated to the horosphere L_ϱ and the hyperplane $\mathcal{L}_\tau$, respectively (see Section 4). In our approach, we use suitable expressions of the Cheng-Yau's square operator [6] acting on the height and support functions of the hypersurface (see Lemma 2.1).

As an application of our characterization of the Euclidean domains of $\mathcal{H}^{n+1}$, we obtain the following rigidity result concerning complete hypersurfaces with *one end* (that is, complete hypersurfaces which can be regarded as the union of a compact hypersurface whose boundary is contained into a hyperplane of $\mathcal{H}^{n+1}$, with a complete hypersurface diffeomorphic to a cylinder; see Theorem 3.4):

The spacelike hyperplanes are the only complete spacelike hypersurfaces with one end in $\mathcal{H}^{n+1}$ *with* H_2 *constant and whose mean curvature* H *satisfies* $1 \leq H \leq H_2$.

Finally, in $\mathbb{H}^{n+1}$ we also establish the following characterization for the horospheres (cf. Theorem 4.2):

Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ *be a complete hypersurface with one end, which is tangent from below at the infinity to a horosphere* L_ϱ *of* $\mathbb{H}^{n+1}$, *for some* $\varrho > 0$. *Suppose that* $N(\Sigma)$ *is contained into* $\mathcal{L}_\tau^+$, *for some* $\tau > 0$. *If* H_2 *is constant and the mean curvature* H *satisfies*

$$H \geq H_2 \geq \left(\frac{\varrho}{\tau}\right)^2,$$

then Σ^n *is a horosphere of* $\mathbb{H}^{n+1}$.

2 Preliminaries

Let $\overline{M}^{n+1}$ be a semi-Riemannian manifold of index $\nu \leq 1$, with metric tensor $\langle\, ,\rangle$ and Levi-Civita conection $\overline{\nabla}$. We denote by $C^\infty(\overline{M})$ the ring of real functions of class C^∞ on $\overline{M}^{n+1}$ and by $\mathfrak{X}(\overline{M})$ the $C^\infty(\overline{M})$-module of vector fields of class C^∞ on $\overline{M}^{n+1}$. For a vector field $X \in \mathfrak{X}(\overline{M})$, let $\epsilon_X = \langle X, X\rangle$. We say that X is a *unit vector field* when $\epsilon_X = \pm 1$, *timelike unit vector field* if $\epsilon_X = -1$ and *spacelike unit vector field* if $\epsilon_X = 1$.

In all that follows, we consider Riemannian immersions $\psi : \Sigma^n \to \overline{M}^{n+1}$, namely, immersions from a connected, n-dimensional orientable differentiable manifold Σ^n into the semi-Riemannian manifold $\overline{M}^{n+1}$, such that the induced metric tensor turns Σ^n into a Riemannian manifold. In this case, it is customary to denote (and so will do) the metric tensors of Σ^n and $\overline{M}^{n+1}$ by the same symbol. In the Lorentz case (that is, $\nu = -1$), we refer to Σ^n as a *spacelike hypersurface* of $\overline{M}^{n+1}$. When $\overline{M}^{n+1}$ is Riemannian (namely, if $\nu = 0$), Σ^n is always assumed orientable, and in the Lorentzian case, we assume that $\overline{M}^{n+1}$ is time-orientable (cf. [22], Lemma 5.32). In any case Σ^n is orientated by the choice of a unit normal vector field N on it. Let ∇ be a Levi-Civita connection of Σ^n.

In this setting, if we let $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denote the corresponding shape operator (or second fundamental form) of Σ^n, then, at each $p \in \Sigma^n$, A restricts to a self-adjoint linear map $A_p : T_p\Sigma \to T_p\Sigma$. Thus, for fixed $p \in \Sigma^n$, the spectral theorem allows us to choose on $T_p\Sigma$ an orthonormal basis $\{E_1, \dots, E_n\}$ of eigenvectors of A_p, with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$, respectively.

Along this paper, we will deal with the first three r-th mean curvatures of Σ^n, namely

$$H = \frac{1}{n}\,\epsilon_N \sum_{i=1}^{n} \lambda_i,$$

$$H_2 = \frac{2}{n(n-1)} \sum_{i<j} \lambda_i \lambda_j,$$

$$H_3 = \frac{6}{n(n-1)(n-2)}\,\epsilon_N \sum_{i<j<k} \lambda_i \lambda_j \lambda_k.$$

One also let the *Newton transformation* $T : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ be given by setting

$$T = nH\,\mathrm{Id} - \epsilon_N A,$$

where $\mathrm{Id} : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the identity map.

Associated to Newton transformation T one has the well known *Cheng-Yau's square operator* [6]

$$\begin{array}{rccl} \Box : & C^\infty(\Sigma) & \to & C^\infty(\Sigma) \\ & \xi & \mapsto & \Box\,\xi = \mathrm{tr}(T \circ \mathrm{Hess}\,\xi). \end{array}$$

When $\overline{M}^{n+1}$ has constant sectional curvature, Rosenberg proved in [23] that

$$\Box\,\xi = \mathrm{div}_\Sigma(T\,\nabla\xi),$$

where div_Σ stands for the divergence on Σ^n. Moreover, from Lemma 3.10 of [9], when $H_2 > 0$ we have that $\Box$ is an elliptic operator.

For a smooth function $g : \mathbb{R} \to \mathbb{R}$ and $\xi \in C^\infty(\Sigma)$, it follows from the properties of the Hessian of functions that

$$\Box(g \circ \xi) = g'(\xi)\Box\,\xi + g''(\xi)\langle T\nabla\xi, \nabla\xi\rangle. \tag{2.1}$$

Now, let M^n be a n-dimensional ($n \geq 2$) oriented Riemannian manifold with metric $\langle\, , \rangle_M$, $I \subset \mathbb{R}$ an interval with the induced metric dt^2 and $f : I \to \mathbb{R}$ a positive smooth function. In the product differentiable manifold $I \times M^n$, let π_I and π_M denote the projections onto the I and M factors, respectively. A particular class of semi-Riemannian manifolds is the one obtained by furnishing $I \times M^n$ with the metric tensor

$$\langle\, , \rangle = \epsilon\,(\pi_I)^*\left(dt^2\right) + (f \circ \pi_I)^2\,(\pi_M)^*\left(\langle\, , \rangle_M\right),$$

where $\epsilon \in \{-1, 1\}$. Such a space is called a *semi-Riemannian warped product*, and in what follows we will write $\epsilon I \times_f M^n$ to denote it. In this setting, M^n is called the *Riemannian fiber* of $\epsilon I \times_f M^n$.

The formulas collected in the following lemma are particular cases of results obtained by Alías and Colares in [2] and Alías, Impera and Rigoli in [4].

Lemma 2.1. *Let $\overline{M}^{n+1} = \epsilon I \times_f M^n$ be a semi-Riemannian warped product and $\psi : \Sigma^n \to \overline{M}^{n+1}$ a Riemannian immersion, with unit normal vector field N. Then, by denoting $h = \pi_I|_\Sigma : \Sigma^n \to I$ the vertical height function of Σ^n, we have*

$$\Box h = (\log f)'(h)\{\epsilon\, n(n-1)H - \langle T\nabla h, \nabla h\rangle\} + \epsilon\, n(n-1)\langle N, \partial_t\rangle H_2.$$

Moreover, if the Riemannian fiber M^n of $\overline{M}^{n+1}$ has constant sectional curvature κ, we have that

$$\begin{aligned}\Box\langle N, f(h)\partial_t\rangle &= -\epsilon\frac{n(n-1)}{2}\{\langle\nabla H_2, f(h)\partial_t\rangle + 2f'(h)H_2\\ &+\langle N, f(h)\partial_t\rangle(nHH_2-(n-2)H_3)\}\\ &-\epsilon\, n(n-1)\langle N, f(h)\partial_t\rangle\left\{\frac{\kappa}{f^2(h)}-(\log f)''(h)\right\}|\nabla h|^2\\ &+\epsilon\,\langle N, f(h)\partial_t\rangle\left\{\frac{\kappa}{f^2(h)}-(\log f)''(h)\right\}\langle T\nabla h,\nabla h\rangle.\end{aligned}$$

3 Euclidean domains of the steaty state space

Let $\mathbb{L}^{n+2}$ denote the $(n+2)$-dimensional Lorentz-Minkowski space ($n \geq 2$), that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentz metric defined by

$$\langle v, w\rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2}w_{n+2},$$

for all $v, w \in \mathbb{R}^{n+2}$. We define the $(n+1)$-dimensional de Sitter space $\mathbb{S}_1^{n+1}$ as the following hyperquadric of $\mathbb{L}^{n+2}$

$$\mathbb{S}_1^{n+1} = \left\{p \in L^{n+2}; \langle p, p\rangle = 1\right\}.$$

The induced metric from $\langle\,,\rangle$ makes $\mathbb{S}_1^{n+1}$ into a Lorentz manifold with constant sectional curvature one. Let $a \in \mathbb{L}^{n+2}$ be a past-pointing null vector, that is, $\langle a, a\rangle = 0$ and $\langle a, e_{n+2}\rangle > 0$, where $e_{n+2} = (0, \ldots, 0, 1)$. Then the open region of the de Sitter space $\mathbb{S}_1^{n+1}$, given by

$$\mathcal{H}^{n+1} = \left\{x \in \mathbb{S}_1^{n+1}; \langle x, a\rangle > 0\right\}$$

is the so-called *steady state space*. Observe that $\mathcal{H}^{n+1}$ is a non-complete manifold, being only half of the de Sitter space. Its boundary, as a subset of $\mathbb{S}_1^{n+1}$, is the null hypersurface

$$\mathcal{L}_0 = \left\{x \in \mathbb{S}_1^{n+1}; \langle x, a\rangle = 0\right\},$$

whose topology is that of $\mathbb{R} \times \mathbb{S}^{n-1}$ (cf. Section 2 of [21]).

According [1], the null hypersurface $\mathcal{L}_0$ represents the *past infinity* of $\mathcal{H}^{n+1}$, while the limit boundary

$$\mathcal{L}_\infty = \{\, x \in \mathbb{S}_1^{n+1}; \langle x, a\rangle = \infty\,\}$$

represents the *future infinity* of $\mathcal{H}^{n+1}$.

Remark 3.1. *We note that $\mathcal{H}^{n+1}$ corresponds to a model of the universe that was proposed by Bondi and Gold [5] and Hoyle [12], when looking for a model of the universe which looks the same not only at all points and in all directions, that is, spatially isotropic and homogeneous, but at all times (cf. Section* 5.2 *of [10]).*

Now, we shall consider in $\mathcal{H}^{n+1}$ the timelike field

$$\mathcal{K} = -\langle x, a \rangle x + a.$$

We easily see that

$$\overline{\nabla}_V \mathcal{K} = -\langle x, a \rangle V, \quad \forall V \in \mathfrak{X}(\mathcal{H}^{n+1}),$$

that is, $\mathcal{K}$ is closed and conformal field on $\mathcal{H}^{n+1}$ (cf. [14], Section 5). Then, from Proposition 1 of [19], we have that the n-dimensional distribution $\mathcal{D}$ defined on $\mathcal{H}^{n+1}$ by

$$p \in \mathcal{H}^{n+1} \longmapsto \mathcal{D}(p) = \left\{ v \in T_p\mathcal{H}^{n+1}; \langle \mathcal{K}(p), v \rangle = 0 \right\}$$

determines a codimension one spacelike foliation $\mathcal{F}(\mathcal{K})$ which is oriented by $\mathcal{K}$. Moreover, from Example 1 of [18], we conclude that the leaves of $\mathcal{F}(\mathcal{K})$ are given by

$$\mathcal{L}_\tau = \left\{ x \in \mathbb{S}_1^{n+1}; \langle x, a \rangle = \tau \right\}, \quad \tau > 0,$$

which are totally umbilical hypersurfaces of $\mathcal{H}^{n+1}$ isometric to the Euclidean space $\mathbb{R}^n$, and having constant mean curvature one with respect to the unit normal fields

$$N_\tau = x - \frac{1}{\tau} a, \quad x \in \mathcal{L}_\tau. \tag{3.1}$$

Remark 3.2. *An explicit isometry between the leaves $\mathcal{L}_\tau$ and $\mathbb{R}^n$ can be found at Section* 2 *of [1]. So, in what follows, we will refer each leave $\mathcal{L}_\tau$ as a hyperplane of $\mathcal{H}^{n+1}$.*

We observe that the steady state space $\mathcal{H}^{n+1}$ can also be expressed in an isometrically equivalent way as the following Robertson-Walker spacetime:

$$-\mathbb{R} \times_{e^t} \mathbb{R}^n.$$

To see it, take $b \in \mathbb{L}^{n+2}$ another null vector such that $\langle a, b\rangle = 1$ and let $\Phi : \mathcal{H}^{n+1} \to -\mathbb{R} \times_{e^t} \mathbb{R}^n$ be the map given by

$$\Phi(x) = \left(\ln(\langle x, a\rangle), \frac{x - \langle x, a\rangle b - \langle x, b\rangle a}{\langle x, a\rangle}\right).$$

Then it can easily be checked that Φ is an isometry between both spaces which conserves time orientation (see [1], Section 4). In particular, for all $\tau > 0$, we have that

$$\Phi\left(\mathcal{L}_\tau\right) = \{\ln \tau\} \times \mathbb{R}^n \quad \text{and} \quad \Phi_*(N_\tau) = \partial_t.$$

Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a spacelike hypersurface of $\mathcal{H}^{n+1}$. We recall that there exists a unique unitary timelike normal field N globally defined on Σ^n which is future-directed (that is, $\langle N, e_{n+2}\rangle < 0$). Throughout this paper we will refer to N as the *future-pointing Gauss mapping* of Σ^n. Moreover, we note that the Gauss mapping N of Σ^n can be thought of as a map

$$N : \Sigma^n \to \mathbb{H}^{n+1}$$

taking values in the hyperbolic space

$$\mathbb{H}^{n+1} = \left\{x \in \mathbb{L}^{n+2}; \langle x, x\rangle = -1,\ \langle x, a\rangle < 0\right\},$$

where a is a non-zero null vector in $\mathbb{L}^{n+2}$, which will be chosen as in the previous section. In this setting, the image $N(\Sigma)$ is called the *hyperbolic image* of Σ^n. Furthermore, we note that all the horospheres of $\mathbb{H}^{n+1}$ can be realized in the Minkowski model in the following way

$$L_\rho = \left\{x \in \mathbb{H}^{n+1}; \langle x, a\rangle = \rho\right\}, \quad \rho > 0.$$

As observed in [7], when Σ^n is a spacelike hypersurface of $\mathcal{H}^{n+1}$ whose boundary in some hyperplane $\mathcal{L}_\tau$ and its hyperbolic image is contained in the closure of the interior domain enclosed by some horosphere L_ϱ, we must have $\varrho \geq \tau$.

When a compact spacelike hypersurface $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ is entirely contained into some hyperplane $\mathcal{L}_\tau$, it is called an *Euclidean domain* of $\mathcal{H}^{n+1}$. Now, we are in position to state and prove our first result.

Theorem 3.3. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a compact spacelike hypersurface with H_2 constant and whose boundary $\partial\Sigma$ is contained in a hyperplane $\mathcal{L}_\tau$, for some $\tau > 0$. If the mean curvature H verifies*

$$1 \leq H \leq H_2, \tag{3.2}$$

then Σ^n is an Euclidean domain of $\mathcal{H}^{n+1}$.

Proof. We consider the warped product model $\mathcal{H}^{n+1} = -\mathbb{R} \times_{e^t} \mathbb{R}^n$ and define the function $\xi : \Sigma^n \to \mathbb{R}$ by

$$\xi(p) = c\, e^{h(p)} - \langle N, V \rangle_p, \tag{3.3}$$

where $h = \pi_{\mathbb{R}}|_{\Sigma} : \Sigma^n \to \mathbb{R}$ is the vertical height function of Σ^n, $V = e^h \partial_t$, N is the future-pointing Gauss mapping of Σ^n and c is a positive constant.

From equation (2.1) and Lemma 2.1 we have

$$\begin{aligned} \Box \xi &= c \Box e^h - \Box \langle N, V \rangle \\ &= -n(n-1) c\, e^h (H + \langle N, \partial_t \rangle H_2) \\ &\quad -\frac{1}{2} n(n-1) e^h \left\{ 2H_2 + \langle N, \partial_t \rangle (nHH_2 - (n-2)H_3) \right\}. \end{aligned} \tag{3.4}$$

We claim that

$$nHH_2 - (n-2)H_3 \;\geq\; 2H_2^{3/2}. \tag{3.5}$$

In fact, from (3.2) and Proposition 2.3 of [9] we have $H \geq H_2^{1/2}$ and $H_2^2 \geq HH_3$. Next,

$$HH_2 - H_3 \;\geq\; HH_2 - \frac{H_2^2}{H} = \frac{H_2}{H}(H^2 - H_2) \;\geq\; 0.$$

Thus,

$$\begin{aligned} nHH_2 - (n-2)H_3 &= nHH_2 - nH_3 + 2H_3 + 2HH_2 - 2HH_2 \\ &= (n-2)(HH_2 - H_3) + 2HH_2 \\ &\geq 2H_2^{3/2}. \end{aligned}$$

Now, since $\langle N, \partial_t \rangle \leq -1$, the relationships given in (3.4) and (3.5) assure us that

$$\begin{aligned} \frac{1}{n(n-1)} \Box \xi &\geq e^h \left\{ -c\, H - c \langle N, \partial_t \rangle H_2 - H_2 - \langle N, \partial_t \rangle H_2^{3/2} \right\} \\ &\geq e^h \left\{ -c\, H + c\, H_2 - H_2 + H_2^{3/2} \right\} \\ &= e^h \left\{ c\,(H_2 - H) + H_2 (H_2^{1/2} - 1) \right\} \;\geq\; 0, \end{aligned}$$

where in last inequality we use hypothesis (3.2). Then $\Box \xi \geq 0$ em Σ^n. Consequently, since $H_2 > 0$ guarantees that the operator $\Box$ is elliptic (see Lemma 3.10 of [9]), the maximum principle ensures that

$$\xi \leq \xi\big|_{\partial \Sigma}. \tag{3.6}$$

On the other hand, the compactness of Σ^n we obtain that the hyperbolic image of Σ^n is contained in the closure of a interior domain of a horosphere $L_\varrho \subset \mathbb{H}^{n+1}$, for some $\varrho > 0$. This implies that $0 > \langle N, a\rangle \geq -\varrho$. Hence, along the boundary $\partial\Sigma$ we have

$$-1 \;\geq\; \langle N, \partial_t\rangle\big|_{\partial\Sigma} \;=\; \langle N, N_\tau\rangle \;=\; \langle N, -\psi + \frac{1}{\tau}a\rangle \;\geq\; -\frac{\varrho}{\tau},$$

where N_τ is the normal field of $\mathcal{L}_\tau \subset \mathcal{H}^{n+1}$. Thus, from (3.3) and (3.6),

$$c\,e^h \;\leq\; c\,e^h - \langle N, e^h\partial_t\rangle \;=\; \xi \;\leq\; \xi\big|_{\partial\Sigma} \;\leq\; c + \frac{\varrho}{\tau}.$$

Thus, we get

$$e^h \leq 1 + \frac{\varrho}{\tau c}. \tag{3.7}$$

Consequently, since the positive constant c is arbitrary, we have that $h \equiv 0$ and, hence, we conclude that $\psi(\Sigma^n) \subset \mathcal{L}_\tau$. □

According [15], we say that a complete spacelike hypersurface $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ has *one end* $\mathcal{C}^n$ if Σ^n can be regarded as $\Sigma^n = \Sigma^n_\tau \cup \mathcal{C}^n$, where Σ^n_τ is a compact hypersurface with boundary contained into a hyperplane $\mathcal{L}_\tau$, for some $\tau > 0$, and $\mathcal{C}^n$ is diffeomorphic to a cylinder $[\ln\tau, +\infty) \times \mathbb{S}^{n-1}$.

From Theorem 3.3, we obtain the following rigidity result:

Theorem 3.4. *The spacelike hyperplanes are the only complete spacelike hypersurfaces with one end in $\mathcal{H}^{n+1}$ with H_2 constant and whose mean curvature H satisfies $1 \leq H \leq H_2$.*

Proof. Suppose, by contradiction that, Σ^n is not a spacelike hyperplane of $\mathcal{H}^{n+1}$. Then there are constants $\tau_2 > \tau_1 > 0$ such that the $\mathcal{L}_{\tau_1} \cap \Sigma^n \neq \emptyset$ and $\mathcal{L}_{\tau_2} \cap \Sigma^n \neq \emptyset$. Consequently, from Theorem 3.3, we get that $\Sigma^n_{\tau_1} \subset \mathcal{L}_{\tau_1}$ and $\Sigma^n_{\tau_2} \subset \mathcal{L}_{\tau_2}$. Hence, since $\Sigma^n_{\tau_1} \subset \Sigma^n_{\tau_2}$, we arrive at a contradiction. □

4 Euclidean domains of the hyperbolic space

In this section, instead of the more commonly used half-space model for the $(n+1)$-dimensional hyperbolic space, we consider the warped product model

$$\mathbb{H}^{n+1} = \mathbb{R} \times_{e^t} \mathbb{R}^n.$$

An explicit isometry between these two models can be found at [3], from where it can easily be seen that the fibers $M_{t_0} = \{t_0\} \times \mathbb{R}^n$ of the warped product model are precisely the horospheres of $\mathbb{H}^{n+1}$. Moreover, these have constant mean curvature 1 if we take the orientation given by the unit normal vector field $N = -\partial_t$ (cf. Section 4 of [20]).

Another useful model for $\mathbb{H}^{n+1}$ is (following the notation of the previous section) the so-called *Lorentz model*, obtained by furnishing the hyperquadric

$$\{p \in \mathbb{L}^{n+2};\, \langle p, p\rangle = -1,\, p_{n+2} > 0\}$$

with the (Riemannian) metric induced by the Lorentz metric of $\mathbb{L}^{n+2}$. In this setting, if $a \in \mathbb{L}^{n+2}$ denotes a fixed null vector as in the beginning of the previous section, we have that a typical horosphere is

$$L_\varrho = \{p \in \mathbb{H}^{n+1};\, \langle p, a\rangle = \varrho\},$$

for some $\varrho > 0$. A straightforward computation shows that

$$\eta_p = p + \frac{1}{\varrho} a \in \mathcal{H}^{n+1}$$

is a unit normal vector field along L_ϱ, with respect to which L_ϱ has mean curvature -1 (cf. [17]). Therefore, any isometry Φ between the warped product and Lorentz models of $\mathbb{H}^{n+1}$ must carry $(\partial_t)_q$ to $\Phi_*(\partial_t) = \eta_{\Phi(q)}$. Thus, it is natural to consider the *Lorentz Gauss mapping* of a hypersurface $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ with respect to N as given by

$$\begin{array}{ccc} \Sigma^n & \to & \mathcal{H}^{n+1} \\ p & \mapsto & -\Phi_*(N_p) \end{array}$$

In order to establish our next result, according to [17], we will consider the following region naturally associated to a horosphere L_ϱ of $\mathbb{H}^{n+1}$:

$$L_\varrho^- = \{\, x \in \mathbb{H}^{n+1}; \langle x, a\rangle \le \varrho \,\};$$

and, according to [21], we will also work with the following region naturally associated to a hyperplane $\mathcal{L}_\tau$ of $\mathcal{H}^{n+1}$:

$$\mathcal{L}_\tau^+ = \{\, x \in \mathcal{H}^{n+1}; \langle x, a\rangle \ge \tau \,\}.$$

Moreover, in a analogous way of the previous section, when a compact hypersurface $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ is entirely contained into some horosphere L_ϱ, it is called an *Euclidean domain* of $\mathbb{H}^{n+1}$.

Theorem 4.1. *Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ be a compact hypersurface, with H_2 constant and whose boundary $\partial\Sigma$ is contained in a horosphere L_ϱ, for some $\varrho > 0$. Suppose that $\Sigma^n \subset L_\varrho^-$ and that the image of its Gauss mapping $N(\Sigma)$ is contained in $\mathcal{L}_\tau^+ \subset \mathcal{H}^{n+1}$, for some $\tau > 0$. If the mean curvature H verifies*

$$H \geq H_2 \geq \left(\frac{\varrho}{\tau}\right)^2, \tag{4.1}$$

then Σ^n is an Euclidean domain of $\mathbb{H}^{n+1}$.

Proof. We consider the warped product model $\mathbb{H}^{n+1} = \mathbb{R} \times_{e^t} \mathbb{R}^n$ and orient Σ^n by choosing a unit normal vector field N such that

$$\langle N, \partial_t \rangle \geq -1. \tag{4.2}$$

We observe that

$$\Sigma^n \subset L_\varrho^- \qquad \text{if, and only if,} \qquad \langle \psi, a \rangle \leq \varrho, \tag{4.3}$$

and

$$N(\Sigma^n) \subset \mathcal{L}_\tau^+ \qquad \text{if, and only if,} \qquad \langle N, a \rangle \geq \tau. \tag{4.4}$$

Thus, since from (4.3) and (4.4) we have that

$$-1 \;\leq\; \langle N, \partial_t \rangle \;=\; \langle N, -\psi - \frac{1}{\langle \psi, a \rangle} a \rangle \;=\; -\frac{\langle N, a \rangle}{\langle \psi, a \rangle} \leq -\left(\frac{\tau}{\varrho}\right),$$

we get

$$-\langle N, \partial_t \rangle \geq \frac{\tau}{\varrho}, \tag{4.5}$$

On the other hand, similarly as in the proof of Theorem 3.3, our assumption on the curvatures H and H_2 guarantees that the inequality (3.5) is valid.

Now, we consider the function $\xi : \Sigma^n \to \mathbb{R}$ given by

$$\xi(p) = c\, e^{h(p)} + \langle N, V \rangle_p, \tag{4.6}$$

where h is the vertical height function of Σ^n, $V = e^h \partial_t$ and $c > 1$ is a constant.

From equation (2.1) and Lemma 2.1, we obtain

$$\begin{aligned} \Box \xi &= c \Box e^h + \Box \langle N, V \rangle \qquad (4.7) \\ &= n(n-1) c\, e^h (H + \langle N, \partial_t \rangle H_2) \\ &\quad - \frac{n(n-1)}{2} e^h \left\{ 2H_2 + \langle N, \partial_t \rangle (nHH_2 - (n-2)H_3) \right\} \\ &\geq n(n-1) e^h \left\{ c\, (H + \langle N, \partial_t \rangle H_2) - H_2 \left(1 + \langle N, \partial_t \rangle H^{1/2}\right) \right\}, \end{aligned}$$

where in last inequality we use (3.5). So, applying (4.2) and (4.5) in (4.7) we obtain

$$\frac{1}{n(n+1)}\ \Box\xi\ \geq\ e^h\left\{c\ (H-H_2)+H_2\left(\frac{\tau}{\varrho}\,H_2^{1/2}-1\right)\right\}\ \geq\ 0\,,$$

for any constant $c>1$, where in last inequality we use (4.1). Then $\Box\xi\geq 0$ em Σ^n. We observe that $\Box$ is elliptic, because H_2 is constant and $H_2>0$. Thus, the maximum principle ensures that

$$\xi\leq\xi\big|_{\partial\Sigma}. \tag{4.8}$$

From (4.6), (4.2), (4.8) and (4.5) follows that

$$e^h(c-1)\ \leq\ \xi\ \leq\ \xi\big|_{\partial\Sigma}\ =\ c+\langle N,\partial_t\rangle\big|_{\partial\Sigma}\ \leq\ c-\left(\frac{\tau}{\varrho}\right),$$

and therefore

$$e^h\leq\frac{c}{c-1}-\frac{\tau}{(c-1)\varrho}\,,$$

for any constant $c>1$. Finally, following the last steps of the proof of Theorem 3.3, we obtain that $\psi(\Sigma^n)\subset L_\varrho$. $\Box$

Similarly to the situation of the steady state space, we say that a complete hypersurface $\psi:\Sigma^n\to\mathbb{H}^{n+1}$ has *one end* $\mathcal{C}^n$ when Σ^n can be regarded as $\Sigma^n=\Sigma^n_\varrho\cup\mathcal{C}^n$, where Σ^n_ϱ is a compact hypersurface with boundary contained into a horosphere L_ϱ and $\mathcal{C}^n$ is diffeomorphic to the cylinder $[\ln\varrho,+\infty)\times\mathbb{S}^{n-1}$.

Now, let $\psi:\Sigma^n\to\mathbb{H}^{n+1}$ be a complete hypersurface with one. We say that Σ^n is *tangent from below at the infinity* to a horosphere L_ϱ, if either Σ^n is a horosphere $L_{\widetilde{\varrho}}$, for some $\widetilde{\varrho}\leq\varrho$ or, for all $\widetilde{\varrho}\leq\varrho$, we have

i. $L_{\widetilde{\varrho}}\cap\Sigma^n\neq\emptyset$;

ii. the compact part of Σ^n is contained in $L^-_{\widetilde{\varrho}}$.

Finally, we are in position to present our last result:

Theorem 4.2. *Let $\psi:\Sigma^n\to\mathbb{H}^{n+1}$ be a complete hypersurface with one end, which is tangent from below at the infinity to a horosphere L_ϱ, for some $\varrho>0$. Suppose that $N(\Sigma)\subset\mathcal{L}^+_\tau$, for some $\tau>0$. If H_2 is constant and the mean curvature H satisfies*

$$H\geq H_2\geq\left(\frac{\varrho}{\tau}\right)^2,$$

then Σ^n is a horosphere of $\mathbb{H}^{n+1}$.

Proof. Suppose that $L_{\widetilde{\varrho}} \cap \Sigma^n \neq \emptyset$, for some $\widetilde{\varrho} \leq \varrho$. Thus,

$$H \geq H_2 \geq \left(\frac{\varrho}{\tau}\right)^2 \geq \left(\frac{\widetilde{\varrho}}{\tau}\right)^2.$$

Hence, from Theorem 4.1, we obtain that $\Sigma^n \subset L_{\widetilde{\varrho}}$. Therefore, from the completeness of Σ^n we conclude that, in fact, $\Sigma^n = L_{\widetilde{\varrho}}$. □

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