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On Representing an Interval Graph Using the Minimum Number of Interval Lengths

M. R. Cerioli* **F. de S. Oliveira** **J. L. Szwarcfiter***

Abstract

The *interval count problem* is that of determining the smallest number of interval lengths required to represent an interval model of a given interval graph or interval order. Despite the large number of studies about interval graphs and interval orders, few results on the interval count problem exist in fact. We provide a short survey about the interval count and related problems.

1 Introduction

The interest in interval graphs and orders comes from both their central role in many applications and purely theoretical questions [9, 11, 17]. They potentially arise in applications for which there are events associated to time intervals corresponding to the duration of the events. Among such applications, as discussed in [15], there are those related to planning, scheduling, archeology, temporal reasoning, medical diagnosis, and circuit design. Furthermore, there are applications not related directly to duration of events in the fields of genetics, physical mapping of DNA and behavioral psychology.

An *order* $(X, \prec)$ is a transitive and irreflexive binary relation $\prec$ on X. An *interval model* is a family of closed intervals of the real line. A graph G is an *interval graph* if there exists an interval model $\mathcal{R} = \{I_v \mid v \in V(G)\}$ such that for all distinct $x, y \in V(G)$, $I_x \cap I_y \neq \emptyset$ if and only if $xy \in E(G)$. An order $P = (X, \prec)$ is an *interval order* if there exists an interval model

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$\mathcal{R} = \{I_x \mid x \in X\}$ such that $x \prec y$ if and only if I_x precedes (is entirely to the left of) I_y. In these cases, $\mathcal{R}$ is an interval model of G (resp. P). Besides, we say that G (resp. P) is the graph (resp. order) corresponding to $\mathcal{R}$.

For some given graph G (resp. order P), we consider the problem of computing the smallest number $IC(G)$ (resp. $IC(P)$) of interval lengths for an interval model of G (resp. P), named in both cases the *interval count problem* [9, 14]. The interval count problem was suggested by Ronald Graham (cf. [14]). There are graphs having arbitrary interval count values. For instance, let G_1 be a graph having a single vertex u_1. For all $k \geq 2$, let G_k be three disjoint copies of G_{k-1} plus a universal vertex u_k. It is easy to show that $IC(G_k) = k$.

Very intuitive statements made regarding the interval count problem were proven not to hold afterwards. Graham, for instance, stated a conjecture that the interval count of a graph decreases by at most one unit when exactly one vertex is removed (cf. [14]). Intuitively speaking, if a graph has an interval model requiring at least k different interval lengths, the operation of removing one interval from this model (or, equivalently, a vertex of this graph) seems not to result in a graph which has an interval model using $k-2$ or less different interval lengths. This conjecture was proven to hold only for certain interval graphs, as we discuss in the next sections.

Since the interval count problem is defined only for interval graphs (resp. orders), it is assumed that the graphs (resp. orders) have an associated interval model when this problem is considered. Denoting by $IC(\mathcal{R})$ the number of distinct lengths of a given interval model $\mathcal{R}$, for a given order P we can write:

$$IC(P) = \min\{IC(\mathcal{R}) \mid \mathcal{R} \text{ is an interval model of } P\}$$

and, similarly, given a graph G:

$$IC(G) = \min\{IC(\mathcal{R}) \mid \mathcal{R} \text{ is an interval model of } G\}$$

When there exists an interval model $\mathcal{R}$ which is an interval model of both an order P and a graph G, we say that P *agrees* with G. Note that an interval order agrees with a unique interval graph, but the converse is false: an interval graph G may have exponentially many interval orders agreeing with G (namely, those obtained from each transitive orientation of the complement of G). Using this relation of agreement, we can formulate the interval count of graphs in such a way as to make it explicit its relationship to the interval count of orders. Given a graph G, we have:

$$IC(G) = \min\{IC(P) \mid P \text{ agrees with } G\}$$

It is clear and well known that an interval graph or an interval order has an interval model in which all interval extreme points are integer numbers. Recently, we [3] considered the question of how the interval count of a graph (order) is affected when the interval models are assumed to have distinct integer extreme points. We showed that it is in fact invariant under such an assumption. Therefore, previous results on interval count problem, to be presented on the following sections, do not consider such an assumption.

Denote the left and right extreme points of an interval I by $\ell(I)$ and $r(I)$, respectively. When an interval model of an interval graph (resp. interval order) is clear in the context, for convenience we may use the concepts of vertex (resp. element) and its corresponding interval interchangeably. A *unit interval graph* is an interval graph which admits an interval model whose intervals have unitary length. A $K_{1,r}$ graph is the complete bipartite graph in which the cardinalities of the sets forming the bipartition are 1 and r. A $(1 + 3)$ *order* is one isomorphic to the order $(\{a, b, c, d\}, \prec)$ such that $b \prec c \prec d$ and a is incomparable to b, c, and d. A *proper interval graph* is an interval graph which admits an interval model for which there are no intervals I_x and I_y such that $\ell(I_x) < \ell(I_y) < r(I_y) < r(I_x)$.

For the omitted notations in this paper, refer to [1] for general graph theory, [20] for general order theory, and [9, 11] for a specialized discussion about interval graphs and interval orders.

2 Interval count one

The question of deciding whether $IC(G) = 1$ for an interval graph G is equivalent to that of recognizing whether G is a unit interval graph. In fact, given an interval model using only intervals of the same length, it is possible to either compress or expand proportionally all intervals so that they are transformed into unit intervals. The recognition problem of unit interval graphs is well-known since the sixties [16] and can be solved by polynomial-time algorithms, some of them being of linear time [5, 6, 7, 8, 10, 12, 16]. Moreover, unit interval graphs are characterized by a single finite forbidden structure, as stated in Theorem 1 (firstly obtained by Roberts [16]).

Theorem 1 (Roberts [16])**.** *If G is an interval graph, then G is a unit interval graph if and only if G is $K_{1,3}$-free.*

Corollary 2 ([5, 6, 7, 8, 10, 12, 16])**.** *If G is an interval graph, then G is a unit interval graph if and only if G is a proper interval graph.*

Corollary 3 (Roberts [16]). *If P is an interval order, then P is a unit interval order if and only if P has no induced* $(1+3)$ *order.*

Therefore, characterizations for graphs and orders having interval count one are well-solved questions. Moreover, note that Graham's conjecture holds trivially for all graphs having interval count one.

3 Interval count two

Skrien [18] characterized the class of graphs that can be represented by a model in which each interval length is either 0 or 1. In spite of that, and fact that deciding whether a graph or order has interval count one is a well-solved problem, it is not known whether the complexity of deciding if $IC(G) = 2$ (resp. $IC(P) = 2$) for a graph G (resp. order P) is a polynomially-time solvable problem.

Fishburn [9] investigated the topology of the models of orders in $\mathcal{P}_2$, the class of orders having interval count equal to 2. Given an order $P \in \mathcal{P}_2$, it is clear that there exist interval models of P having the smallest of the two distinct interval lengths equal to one. The question was to determine the set $\theta(P)$ of admissible lengths for the greatest length. In other words, given an interval order $P = (X, \prec)$, the problem is to determine the set:

$$\theta(P) = \{\ \alpha > 1 \mid \text{ there exists an interval model } \mathcal{R} = \{I_x \mid x \in X\} \text{ of } P \text{ having } IC(\mathcal{R}) = 2 \text{ such that } |I_x| \in \{1, \alpha\} \text{ for each } x \in X\}$$

As an example, if $\mathcal{R}$ is an interval model of the graph $K_{1,t+2}$, $t \geq 1$, and P is the corresponding interval order to $\mathcal{R}$, then $\theta(P) = (t, \infty)$. In a first glance, $\theta(P)$ seems to be continuous. Moreover, in a two-length interval model, it seems to be possible to increase the longer lengths by some small amount without affecting the interval count of the model, as it is possible in the case of a $K_{1,3}$'s model. Fishburn proved that, for some orders, such an increase on the longer length has a limit. He presented examples of orders $P \in \mathcal{P}_2$ such that $\theta(P) = (1, k)$ for each $k \geq 2$. Figure 1 presents such an example for $k = 2$. Examples for $k > 2$ can be found in [9].

Trotter [19] conjectured that $\theta(P)$ would be an open interval. However, Fishburn also presented orders $P \in \mathcal{P}_2$ such that $\theta(P) = (2-1/k, 2) \cup (k, \infty)$ for each $k \geq 2$, which means that surprisingly $\theta(P)$ can be a discontinuous set. Furthermore, he proved that for each $k \geq 2$, there exists $P \in \mathcal{P}_2$ such that $\theta(P)$ is the union of k distinct open intervals.

Regarding Graham's conjecture, all graphs having interval count two trivially satisfy it. Furthermore, Leibowitz, Assmann, and Peck [14] characterized another case for which the conjecture holds: if G is a graph such

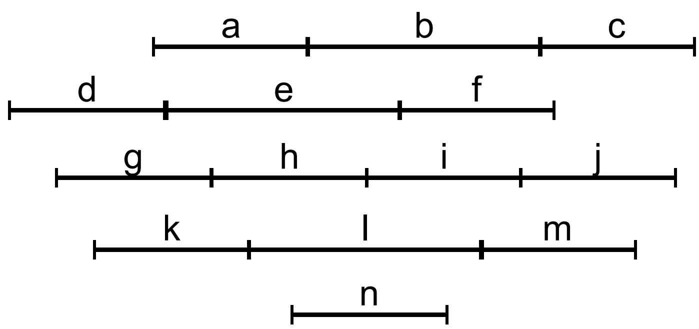

Figure 1: Example of order P having $\theta(P) = (1, 2)$.

that $IC(G \setminus x) = 1$ for some vertex x of G, then $IC(G) \leq 2$.

4 Arbitrary interval count

Like the complexity of deciding the existence of an interval model using two interval lengths, currently it is not known whether deciding if the interval count of graph is k, for an integer $k > 1$, is an NP-complete problem.

Obviously, $IC(G) \leq |V(G)|$ for any graph G, which consists in a trivial upper bound on the interval count of a graph. A better upper bound can be derived by the following observation. Let $\mathcal{R}$ be an interval model of the graph G such that $IC(\mathcal{R}) = IC(G)$. Reading the maximal cliques of G from left to right in $\mathcal{R}$, note that there exists an interval I_1 belonging exclusively to the first maximal clique, or otherwise the first maximal clique would be a subset of the second one. By a similar argumentation, there exists an interval I_q belonging exclusively to the last maximal clique. Since there is no reason to have I_1 and I_q with distinct lengths, we have $IC(G) \leq |V(G)| - 1$ for any graph G. In fact, it can be shown that if a graph G has q maximal cliques, then $IC(G) \leq \lfloor (q+1)/2 \rfloor$. Since in general $q \leq |V(G)|$, then $IC(G) \leq \lfloor (n+1)/2 \rfloor$ as well.

Since having interval count one has a simple characterization in terms of forbidden induced subgraphs, a natural approach is to investigate characterizations of graphs (resp. orders) of arbitrary interval count values by forbidden induced subgraphs (resp. suborders) as well. Fishburn [9] showed that the list of forbidden suborders to characterize the orders which have interval count equal to $k \geq 2$ is infinite. A similar result holds for graphs.

In contrast to the previous sections, Graham's conjecture does not hold for graphs having arbitrary interval count values. Leibowitz, Assmann, and Peck [14] presented examples of graphs having $IC(G) > 2$ for which $IC(G) = IC(G \setminus x) + 2$ given a particular vertex x of G. Trotter [19] conjectures that the removal of a vertex may decrease the interval count by an arbitrary large amount.

Fishburn [9] also considered extremal problems on the interval count of an order, its number of elements, and its number of maximal antichains. Given a graph G (resp. order P), denote by $q(G)$ (resp. $q(P)$) the number of its maximal cliques (resp. maximal antichains). Consider the following functions:

$\sigma(k) = \min\{|X| \mid P = (X, \prec)$ is an order and $IC(P) \geq k\}$; and

$\nu(k, q) = \min\{|X| \mid P = (X, \prec)$ is an order, $q(P) = q$, $IC(P) \geq k\}$

Obviously, $\sigma(k)$ is equal to the minimum of $\nu(k, q)$ varying q over its domain. Particularly, Fishburn showed that $\sigma(k) = \min\{\nu(k, q) \mid q \geq 2k - 1\}$. The restriction $q \geq 2k - 1$ follows from the fact that the function $\nu(k, q)$ is undefined for each $k > \lfloor (q + 1)/2 \rfloor$, i.e. $IC(P) < k$ for all $k > \lfloor (q + 1)/2 \rfloor$ and order P such that $q(P) = q$. Besides he proved that $\nu(k, q) \leq k + q - 1$ holds in general and calculated the exact value of the function $\nu(k, q)$ when k and q are restricted to some specific values.

5 Restricting graphs and orders to subclasses

In this section we present results which come from the investigation of the interval count problem restricted to certain subclasses of interval graphs and interval orders.

A P_n is an induced path on n vertices. Let G be a graph and $v \in V(G)$. The *neighborhood* of v is the set $N(v) = \{w \mid (v, w) \in E(G)\}$. The *substitution* of v by the graph G' is the graph H obtained from the disjoint union $(G \setminus v) \cup G'$ plus the edges uw such that $u \in N(v)$ and $w \in V(G')$. In such a case, we say that H is obtained from G by *substituting* v by G'.

A graph is a *tree* if it is connected and acyclic. A graph is *threshold* if its vertex set can be partitioned into $K \cup I$ such that K is a clique, I is an independent set and there exists an ordering $v_1, \ldots, v_{|I|}$ of the vertices of I such that $N(v_i) \subseteq N(v_{i+1})$ for each $1 \leq i < |I|$ (or, equivalently, there exists an ordering $u_1, \ldots, u_{|K|}$ of the vertices of K such that $I \cap N(u_i) \subseteq I \cap N(u_{i+1})$ for each $1 \leq i < |K|$). A graph G is *almost-$K_{1,3}$-free* if there exists $v \in V(G)$ such that $G \setminus v$ is $K_{1,3}$-free. A graph G is *starlike-threshold* if it can be obtained from a threshold graph substituting each vertex of the independent set by a corresponding clique. An interval graph is *trivially perfect* (TP) if it is P_4-free [11].

A graph is *generalized-threshold* if it can be obtained from a threshold graph by substituting each vertex of the independent set by a corresponding unit-interval graph. An XF_1^n graph ($n \geq 0$) consists of a path P of length $n+3$ and a vertex that is adjacent to every vertex of P except its extremes [2]. Therefore, XF_1^0 is a $K_{1,3}$, and XF_1^1 is a bull. For convenience, in this paper we call the graph XF_1^n for each $n \geq 1$ an *extended-bull*. The extended-bull graph is depicted in Figure 2. A graph is *extended-bull-free* if it has no extended-bull as an induced subgraph.

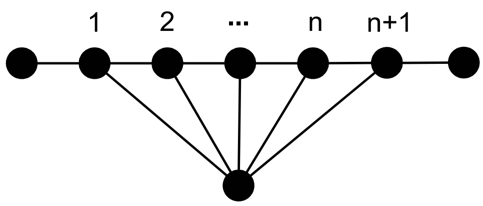

Figure 2: The extended-bull graph, for $n \geq 1$.

Leibowitz [13] proved that the interval count of trees, threshold graphs and almost-$K_{1,3}$-free graphs is at most 2. Cerioli and Szwarcfiter [4] observed that the starlike-threshold graphs also have interval count at most 2. Recently, Cerioli, Oliveira and Szwarcfiter [3] extended the property of having the interval count limited to 2 to generalized-threshold graphs. Furthermore, in [3], polynomial-time algorithms have been described to compute the interval count of extended-bull-free graphs (and, in particular, trivially perfect graphs). Such a class contains instances of graphs with arbitrary interval count values.

To our knowledge, there are no other subclasses of interval graphs and orders for which it is currently known how to compute their interval count efficiently.

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Universidade Federal do Rio de Janeiro
COPPE
Caixa Postal 68511
Rio de Janeiro - Brazil
cerioli@cos.ufrj.br, fabsoliv@cos.ufrj.br, jayme@nce.ufrj.br