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Even pairs in planar Berge graphs

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Abstract

An even pair in a graph is a pair of vertices such that every induced path between them has even length. It is known that deciding the existence of an even pair in a graph is co-NP-complete [3]. In 1990, Reed conjectured that this problem is polynomial time solvable for perfect graphs [22]. In 2005, Chudnovsky et al. obtained an $O(n^9)$ time algorithm to recognize perfect graphs [4]. As a direct consequence, they obtained an $O(n^{11})$ time algorithm to decide if a perfect graph contains an even pair, proving Reed's conjecure. Even when restricted to planar perfect graphs, this was also the best known algorithm. In this paper, we characterize even pairs in planar perfect graphs. Our characterization leads to an $O(n^3)$ time algorithm to decide the existence of an even pair (and to find it if it exists) in this class.

1 Introduction

We say that a simple graph is Berge if it contains no odd hole and no odd anti-hole, where a hole is a chordless cycle with at least four vertices and an anti-hole is the complementary graph of a hole. Claude Berge called a graph G *perfect* if, for every induced subgraph H of G, the clique number of H is equal to its chromatic number [2]. Berge also conjectured that a graph is perfect if and only if it is a Berge graph. This conjecture is called the *Strong Perfect Graph Conjecture* and it was proved in 2006 by Chudnovsky, Robertson, Seymour and Thomas [5]. In 2009, Chudnovsky and Seymour

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presented a shorter proof of the Strong Perfect Graph Theorem (removing about 50 pages) using the concept of *even pair* [6].

An *even pair* in a graph is a pair of non-adjacent vertices such that every induced path between them has even length. Even pairs were introduced as a tool to characterize minimal imperfect graphs [21] as well as to color perfect graphs, since the set of perfect graphs is closed under even pair contractions [12].

We say that a graph is *perfectly contractile* if, for every induced subgraph, there is a sequence of even pair contractions that produces a complete graph. In that case, the size of the complete graph is the clique number and the chromatic number of the original graph.

Everett and Reed [11] conjectured that a graph is perfectly contractile if and only if it contains no odd hole, no anti-hole and no odd stretcher, where an odd stretcher is a graph with two triangles and three vertex disjoint induced paths joining them, each path having odd length (for example, $\overline{C_6}$). Everett-Reed's conjecture was proved true for several graph classes. For example, planar graphs [19], claw-free graphs [17], bull-free graphs [8], diamond-free graphs [23] and special square-free graphs [18]. A general overview on this results can be found in [10].

Unfortunately, deciding the existence of an even pair in a general graph is co-NP-complete [3]. Also, up to now, there is no polynomial time algorithm to find even pairs in perfectly contractile graphs. In 2006, for example, Figueiredo, Maffray and Maciel proved that every bull-reducible Berge graph or its complement contains an even pair [9], but they do not provide a polynomial time algorithm to find it.

In 1990, Reed conjectured that deciding the existence of an even pair is polynomial time solvable when restricted to the class of Berge graphs [22]. In 2005, Chudnovsky et al. [4] obtained an $O(n^9)$ time algorithm to decide if a graph is Berge. This result proves Reed's conjecture, since the following procedure can be used to decide if a pair of non-adjacent vertices $\{x, y\}$ is an even pair in a Berge graph: include a new vertex z and two new edges xz and yz. It is not hard to see that this new graph is Berge if and only if

$\{x, y\}$ is an even pair in the original Berge graph. This leads to an $O(n^{11})$ time algorithm to find even pairs in Berge graphs.

In [16], Hsu obtained an $O(n^3)$ time algorithm to decide if a planar graph is Berge. One could say that the algorithm briefly described above, using this time Hsu's algorithm, would generate an $O(n^5)$ time algorithm to find even pairs in planar Berge graphs. However, it would not work since including the new vertex and its adjacencies would eventually destroy the planarity of the original graph.

In this paper, we use Hsu's recognition algorithm to obtain an $O(n^3)$ time algorithm to find even pairs in planar Berge graphs, proving the following theorem:

Theorem 1.1. *Given a planar Berge graph G, deciding the existence of an even pair in G is $O(n^3)$ time solvable. If G contains an even pair, we can find one in $O(n^3)$ time.*

Our algorithm is presented in Section 4. To prove Theorem 1.1, we first characterize planar Berge graphs that contain even pairs. In Section 2, we introduce the main ideas of Hsu's algorithms for decomposing planar Berge graphs, as well its terminology [16].

It is interesting to notice that Linhares Sales, Maffray and Reed [19] proved that every planar Berge graph with no odd stretcher is perfectly contractile. For such graphs, it is not difficult to find an even pair (see section 4.5.1 of [1] for an informal discussion). If the graph is drawn in the plane and every face is a triangle, then, according to Hsu's decomposition tree for planar Berge graphs, the graph is a comparability graph and we can find an even pair in polynomial time [7]. If there is a nontriangular face, then they proved that this face contains an even pair at distance two.

However, for planar Berge graphs with odd stretchers, it is not too clear how even pairs can be found quickly. For example, Figure 1 shows a planar Berge graph with only one even pair, represented in black. It is easy to see that this pair remains the unique if we increase the middle part of this graph.

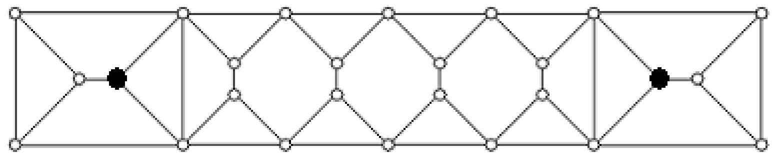

Figure 1: Planar Berge graph with odd stretchers and one even pair (in black)

In another paper, Linhares Sales, Maffray and Reed [20] obtained a polynomial time algorithm to recognize planar strict quasi-parity graphs (planar graphs such that every induced subgraph is complete or contains an even pair). However, they also do not provide an algorithm to find an even pair. Their algorithm uses a graph theoretical characterization of minimal planar non-strict quasi-parity graphs.

The proof of Theorem 1.1 which leads to our $O(n^3)$ polynomial time algorithm is presented in Section 3.

2 Decomposing planar Berge graphs

In this section, we briefly describe Hsu's decomposition for planar graphs. Hsu's decomposition tree can be obtained in $O(n^3)$ time and it is used to decide if a planar graph is perfect.

Given a planar graph $G = (V, E)$, Hsu [16] decomposes G looking for cutsets of G with at most four vertices that satisfies some conditions. By applying these conditions, we can classify the cutsets into seven types: I, IIa, IIb, IIc, IIIa, IIIb and IV, where cutsets of *type I* have only one vertex, cutsets of *type IIa, IIb and IIc* have two vertices, cutsets of *type IIIa and IIIb* have three vertices and cutsets of *type IV* have four vertices. These cutsets are called Hsu-cutsets and are explained below. A cutset with $k+1$ vertices is applied only if all cutsets with at most k vertices were found ($k = 1, 2, 3$).

By applying successively these cutsets, Hsu obtains a decomposition tree T such that the root is G and the nodes of T are obtained from induced subgraphs of G (usually the connected components of a cutset) by adding new vertices and edges in order to keep the parity path properties of the original graph.

Roughly speaking, the decomposition tree can be described as follows: for each node H of T, if H does not have an appropriate cutset, then H is leaf of T. For otherwise, let Q be a Hsu-cutset of H. Every child of H in T is a connected component of $H - Q$ together with Q and possibly with at most three new vertices and edges. Such new vertices are called *artificial* vertices.

More especifically, let $A_1, \dots, A_k$ be the connected components of $H-Q$. For $i = 1, \dots, k$, the children of H will be the graphs B_i which are subgraphs of H induced by the vertices of $V(A_i) \cup Q$ and eventually completed with news edges and artificial vertices.

A Hsu-cutset Q should satisfy one of following conditions:

- *Type I*: Q is only one vertex. The i-th child of H is B_i.
- *Type IIa*: $Q = \{a, b\}$ where ab is an edge. The i-th child of H is B_i.
- *Type IIb*: $Q = \{a, b\}$ where ab is not an edge and there is an induced even $\{a, b\}$-path in H. The i-th child of H is B_i with an artificial vertex e_1 and edges e_1a and e_1b.
- *Type IIc*: $Q = \{a, b\}$ where ab is not an edge and there is an induced odd $\{a, b\}$-path in H. The i-th child of H is B_i with two artificial vertices e_1 and e_2 and edges e_1a, e_2b and e_1e_2.
- *Type IIIa*: $Q = \{a, b, c\}$ where Q is a 3-clique. The i-th child of H is B_i.
- *Type IIIb*: $Q = \{a, b, c\}$, where ab is an edge, ac is not an edge and there is an induced even $\{a, c\}$-path avoiding $\{b\}$ and an induced odd $\{b, c\}$-path avoiding $\{a\}$ in each B_i. The i-th child of H is B_i with an artificial vertex e_1 with edges e_1a and e_1c and possibly a second artificial vertex e_2 with edges e_2b and e_1e_2 if bc is not an edge.
- *Type IV*: $Q = \{a, b, c, d\}$, where Q induces a C_4 (a cycle with four vertices) with edges ab, bc, cd, da and there is an induced even $\{a, c\}$-

path avoiding $\{b, d\}$ and an induced odd $\{b, d\}$-path avoiding $\{a, c\}$ in each B_i. The i-th child of H is B_i with two artificial vertices e_1 and e_2 with edges e_1a, e_1b, e_1c, e_1e_2, e_2a, e_2c and e_2d.

A cutset can be only applied if none of the children of H is isomorphic to H (this isomorphism is easy to check [16]). We say that a graph is k-inseparable if there is no Hsu-cutset with at most k vertices. Therefore, the leaves of T are the 4-inseparable nodes.

Hsu [16] proved that a planar graph G is perfect if and only if every leaf of T belongs to $C \cup L \cup S \cup \{K_3, K_4\}$, where C, L and S are graph classes defined below:

- *Class C*: planar comparability graphs containing an independent subset of C_4-vertices whose deletion produces a bipartite graph, where a C_4-vertex is a vertex whose neighborhood induces a C_4;

- *Class L*: planar line-graphs of bipartite graphs where every vertex has degree 2, 3 or 4 and belongs to exactly two edge-disjoint maximal cliques; and

- *Class S*: graphs S_1, S_2 or S_3 of Figure 2, and any of the five non-isomorphic graphs obtained from S_1 by replacing one or more of the edges e, f, g by a chordless path of length three.

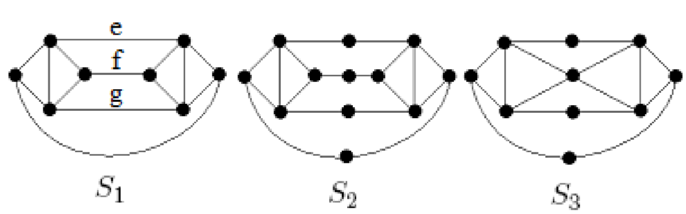

Figure 2: Graphs S_1, S_2 and S_3 of class S

Theorem below from Hsu [16] shows how we can find an even pair in a leaf of class C or class L.

Theorem 2.1 (Theorems 9.2 and 9.3 from [16]). *Let G a graph of class C and let J the set of independent C_4-vertices whose deletion produces a bipartite graph. Then every pair of vertices in $G - J$ is an even pair or an odd pair or an edge. Moreover, let G be a 3-inseparable graph of class L. Then $\{x, y\}$ is an even pair of G if and only if x and y have a common neighbor of degree 2.*

3 Characterizing even pairs in planar Berge graphs

We start by proving the following lemma.

Lemma 3.1. *Let G be a connected planar Berge graph and let T be its Hsu's decomposition tree. Suppose that T satisfies one of the following conditions:*

- *T contains a node H with cutset of type I, IIb, IIIb or IV; or*
- *T contains a leaf $H \in S \cup C$; or*
- *T contains a leaf $H \in L$ which contains a pair of non-artificial vertices with a common neighbor of degree 2.*

Then G contains an even pair that can be found in $O(n^3)$ time.

Proof. It is easy to see that an even pair of non-artificial vertices in a node H of T is also an even pair of G. If a node H has a cutset $Q = \{a\}$ of type I, then any pair $\{x, y\}$ of vertices adjacent to a in different components of $H - Q$ is an even pair. It also follows directly from the definitions that, if H has a cutset Q of type IIb, IIIb or IV, then H has an even pair of vertices belonging to Q. So, suppose that T does not contain a Hsu-cutset of types I, IIb, IIIb or IV.

Now let H be a leaf of T belonging to S. It follows from the definition of class S (see Figure 2) that two non-adjacent vertices in a diamond of H form an even pair. Such vertices are non-artificial, since Hsu's artificial vertices introduced by cutsets of type IIc have degree two.

Now suppose that H belongs to $\mathcal{C}$. Let J be the set of independent C_4-vertices of H whose deletion leaves the graph bipartite. If J is empty, then H is a bipartite graph and, by Theorem 2.1, every pair of non-artificial vertices at distance two is an even pair (clearly H contains a pair satisfying this).

If J is not empty, let $v \in J$ whose neighborhood is $\{a, b, c, d\}$ with edges ab, bc, cd and da. Clearly $a, b, c, d \notin J$. So, by Theorem 2.1, $\{a, c\}$ and $\{b, d\}$ are even pairs. □

This lemma leads us to introduce the notion of *introversive* graphs. A graph is introversive if it is a non-complete connected planar Berge graph which does not satisfy none of the conditions of Lemma 3.1. By consequence of this definition, if G is an introversive graph, then G is not complete, every cutset of its Hsu's decomposition tree T is of type IIa, IIc or IIIa, every leaf of T belongs to $\mathcal{L} \cup \{K_3, K_4\}$ and, if $T \in \mathcal{L}$, any non-artificial vertex u of T is such that $N(u) \cup \{u\}$ induces one of the graphs of Figure 3, where $N(u)$ is the set of neighbours of u.

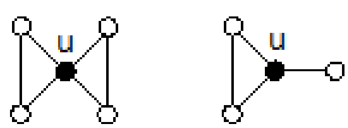

Figure 3: Closed neighboorhood of a non-artificial vertex u of a $\mathcal{L}$-leaf of T

Our main work is to identify an even pair of an introversive graph which was separated by a cutset IIa, IIc or IIIa. Lemmas 3.2 and 3.3 below prove an important result about this question. We postpone their proofs to Section 5.

Let H be an inner node of T and let Q be its cutset. Let x, y be non-artificial vertices in different components of $H - Q$. Let H_X and H_Y be the children of H that contains x and y, respectively.

Let us introduce the following notation. We denote by $P(a, b, c, D)$ the set of all induced paths from a vertex a to a vertex b avoiding a vertex c in a graph D. Observe that, if $P(a, b, c, D)$ is empty, then $\{c\}$ is a cutset of D.

Lemma 3.2. *Suppose that H is 1-inseparable, $\{x, y\}$ is an even pair and H_X and H_Y do not contain even pairs. If Q is of type IIa, then all paths in $P(x, a, b, H_X)$ and $P(x, b, a, H_X)$ have odd length. If Q is of type IIc, then all paths in $P(x, a, b, H_X)$ and $P(x, b, a, H_X)$ have the same parity and $H - Q$ has exactly two components.*

Now, let $P(a, b, c, d, E)$ be the set of all induced paths from a vertex a to a vertex b avoiding vertices $\{c, d\}$ in a graph E. Observe that if $P(a, b, c, d, E)$ is empty, then $\{c, d\}$ is a cutset of E.

Lemma 3.3. *Suppose that H is 2-inseparable, $\{x, y\}$ is an even pair and H_X and H_Y do not contain even pairs. If $Q = \{a, b, c\}$ is of type IIIa, then all paths in $P(x, a, b, c, H_X)$, $P(x, b, a, c, H_X)$ and $P(x, c, a, b, H_X)$ have odd length.*

With Lemmas 3.2 and 3.3, we have the key to deal with the case where a cutset Q of H separates an even pair $\{x, y\}$. The main idea is to include a new artificial vertex z, that we call Z-vertex, in each children B_i of H. The goal is to identify both $\{x, z\}$ and $\{z, y\}$ as even or odd pairs in the children of H and then conclude that $\{x, y\}$ is an even pair of H.

Let H be a non-leaf node of T and let Q be its cutset. The inclusion of the z-vertices will obey the following schema. If $Q = \{a, b\}$ is of type IIa, then we add an artificial Z-vertex z and join it to a and b. If $Q = \{a, b\}$ is of type IIc, then we add an artificial Z-vertex z and join it to a and the Hsu-artificial neighbor of a. Finally, if $Q = \{a, b, c\}$ is a cutset of type IIIa, then we add an artificial Z-vertex z and join it to a, b and c.

From now on, we consider that T is Hsu's decomposition tree of G with Z-vertices. When we say that a leaf of T is of class L, for example, we are considering this leaf without its Z-vertices. As an example, Figure 5 shows the even pair $\{x, y\}$ of the graph of Figure 1 and shows how it can be recovered from the leaves of T, by finding the even or odd pairs $\{x, z_1\}$, $\{z_1, z_2\}$, $\{z_2, z_3\}$, $\{z_3, z_4\}$, $\{z_4, z_5\}$ and $\{z_5, y\}$ in the children of H.

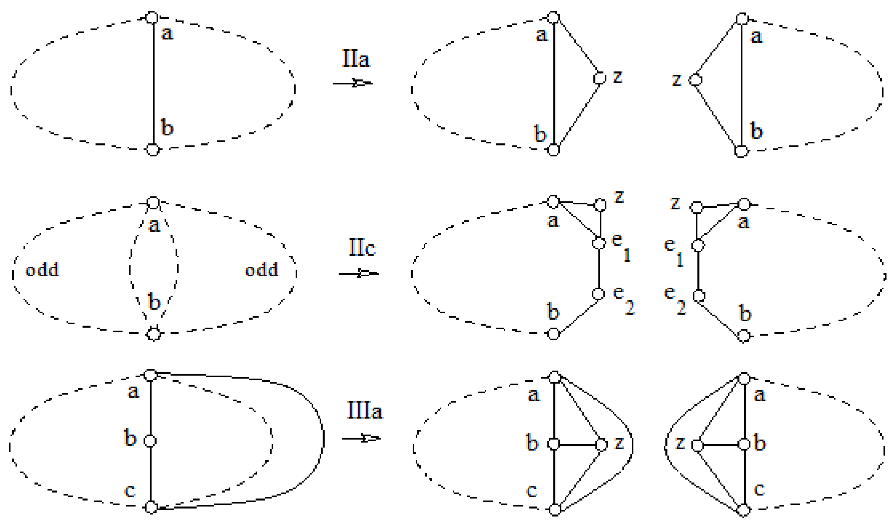

Figure 4: The artificial Z-vertices z of cutsets IIa, IIc and IIIa

With the inclusion of Z-vertices, we expect to track even pairs $\{x, y\}$ that have been separated in the decomposition process. To do so, we have to introduce the definition of $H(x, y)$, $Q(x, y)$ and $Z(x, y)$.

Given vertices x, y of G, we define $H(x, y)$ to be the minimal sequence $(H_0, H_1, \ldots, H_k)$ of leaves of T such that $\cup_{i=0}^{k} V(H_i)$ contains every vertex in a minimum path between x and y in G. We consider that $H(x, y)$ is ordered, that is, H_i is closer to x than H_{i+1}, and H_{i+1} is closer to y than H_i. Let $Q(x, y)$ be the sequence $(Q_1, \ldots, Q_k)$ of cutsets such that Q_i separates H_i from H_{i+1}. Let $Z(x, y)$ be the sequence $(z_1, \ldots, z_k)$ of Z-artificial vertices, where z_i is the Z-vertex introduced by Q_i.

The existence and unicity of $H(x, y)$ are guaranteed by the fact that G is connected and the existence of another $H'(x, y)$ would imply that some cutset separating two consecutives leaves of $H(x, y)$ is not a cutset of G.

Figure 5 shows $H(x, y)$ with six leaves $(H_0, H_1, H_2, H_3, H_4, H_5)$ for the graph of Figure 1. In this example, $Z(x, y) = (z_1, \ldots, z_5)$ and $Q(x, y) = (Q_1, \ldots, Q_5)$ has five cutsets of types IIa, IIc, IIc, IIc and IIa, respectively, where $Q_i = \{a_i, b_i\}$ for $i = 1, \ldots, 5$.

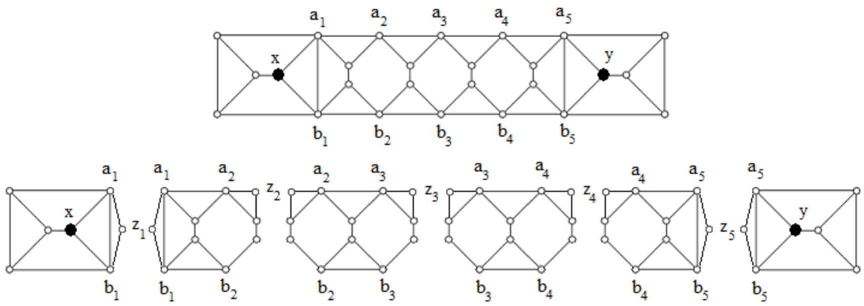

Figure 5: Introversive graph of Figure 1 and its leafs

Definition 3.4. *An even pair $\{x, y\}$ of an introversive graph G is strong if there is no even pair $\{x', y'\}$ such that $x' \in \{H_0, \dots, H_k\}$ and $y' \in \{H_1, \dots, H_{k-1}\}$, where $H(x, y) = \{H_0, \dots, H_k\}$.*

Notice that, if G has an even pair, then G has a strong even pair. We will look for strong even pairs using Z-vertices. The following three lemmas prove important properties of strong even pairs. Their proofs use the following claims whose proofs are in Section 5.

Claim 1. *Let Q be a cutset of G that separates an even pair $\{x, y\}$. Suppose that every path from x and y to any vertex of Q avoiding the others is odd. If Q is of type IIIa, then there exists a vertex b of Q such that either $\{x, b\}$ or $\{y, b\}$ is an edge or an odd pair. If $Q = \{a, b\}$ is of type IIa, then the pairs in one of the following four groups is an edge or an odd pair: (1) $\{x, a\}$ and $\{y, a\}$; (2) $\{x, b\}$ and $\{y, b\}$; (3) $\{x, a\}$ and $\{x, b\}$; or (4) $\{y, a\}$ and $\{y, b\}$.*

Claim 2. *Let H be a leaf of T that, up to the Z-vertices, belongs to class L. Let $\{a, b, c\}$ be a triangle of H. If H does not contain an even pair of non-artificial vertices, then, for every non-artificial vertex u of H, there exists two paths with different parities in each set $P(u, a, b, c, H)$, $P(u, b, a, c, H)$ and $P(u, c, a, b, H)$.*

Using these claims, Lemma 3.5 below proves that strong even pairs satisfy an important property.

Lemma 3.5. *Let $\{x, y\}$ be a strong even pair of G. Let $H(x, y) = \{H_0, \dots, H_k\}$ and $Q(x, y) = \{Q_1, \dots, Q_k\}$ be as defined. If Q_1 (resp. Q_k) is of type IIa or IIIa, then H_0 (resp. H_k) has a vertex adjacent to every vertex of Q_1 (resp. Q_k).*

Proof. If Q_1 is of type IIIa, then, by Lemma 3.3 and Claim 2, H_0 does not belong to class L. Therefore, H_0 is a K_4, and we are done. If $Q_1 = \{a, b\}$ is of type IIa, then, by Claim 1, (i) either $\{x, a\}$ or $\{x, b\}$ is an edge or an odd pair; or (ii) $\{y, a\}$ and $\{y, b\}$ are edges or odd pairs.

Suppose that case (ii) is true. Let M be an induced odd path from y to a avoiding b. Since $\{y, b\}$ is an edge or an odd pair, the path $M + (a, b)$ from y to b cannot be an induced one. Hence there exists an edge joining a vertex m' of M to b. So, the induced path $N = M[y, m'] + m'b$ is an induced path from y to b avoiding a and consequently N is odd. Clearly, the path $N + (b, a)$ from y to a also cannot be an induced one. So, there exists an edge joining N to a. Since M is induced, the only possibility is the edge (m', a). So, $\{x, m'\}$ is an even pair. If $k > 1$, then we have a contradiction, since $\{x, y\}$ is a strong even pair. If $k = 1$, then $Q_1 = Q_k$ and we are done, since we found a vertex m' adjacent to a and b in H_k (the same argument follows for x and H_1).

Now, suppose we have case (i). Let $\{x, a\}$ be an edge or an odd pair. Since H_0, up to the Z-vertices, belongs to class L, then by Theorem 2.1, (x, a) is an edge. Recall that $\{x, y\}$ is a strong even pair. Therefore, by Lemma 3.2, every path from x to b avoiding a is odd. Since H_0 is 3-inseparable, there are at least two vertex-disjoint induced paths U and V from x to b avoiding a.

We have that a must see vertices r of U and s of V, for otherwise $U + xa + ab$ and $V + xa + ab$ would induce an odd hole. By the required neighborhood of graphs in Class L (Figure 3), b must be adjacent to r or s. Therefore, either r or s is adjacent to every vertex of Q_0. The same arguments can be used to Q_k. $\square$

Lemma 3.6 below shows that only the cutsets Q_1 and Q_k can be of type

IIa or IIIa.

Lemma 3.6. *Let $\{x, y\}$ be a strong even pair of G such that $H(x, y) = \{H_0, \ldots, H_k\}$ and $Q(x, y) = \{Q_1, \ldots,$
$Q_k\}$. Then every cutset in $\{Q_2, \ldots, Q_{k-1}\}$ is of type IIc.*

Proof. By contradiction, suppose that there exists a cutset Q_i $(0 < i < k-1)$ of type IIa or IIIa. By Claim 1, there exists a vertex $b \in Q_i$ such that $\{x, b\}$ or $\{y, b\}$ is an odd pair. Without lost of generality, suppose that $\{x, b\}$ is an odd pair. By Lemmas 3.2 and 3.3, every induced path from x to a vertex of Q_i avoiding the others is odd. Since $\{x, b\}$ is an odd pair, there is no induced odd path from x to b containing another vertex of Q_i. Therefore, every induced path from x to b contains a neighbor b_0 of b that does not belong to Q_i. If a neighbor $b_0 \notin Q_i$ of b is adjacent to another vertex of $Q_i - \{b\}$, then the only possibility for the neighborhoods of Figure 3 (recall that H_i belongs to class L) is that Q_i is of type IIa and, in this case, $\{b_0, y\}$ is an even pair, contradicting the fact that $\{x, y\}$ is a strong even pair.

As a consequence, every neighbor $b_0 \notin Q_i$ of b is non-adjacent to every vertex of $Q_i - \{b\}$. We have that one or two vertices satisfy this property (see Figure 3). If there is only one, then this vertex is a cutset of type I, separating b from other vertices of H_i.

So, let $b_1, b_2 \notin Q_i$ be the neighbors of b which are non-adjacent to every vertex of $Q_i - \{b\}$. Clearly, (b_1, b_2) is an edge (see Figure 3) and $\{b_1, b_2\}$ is a cutset of type IIa which separates x from the vertices of Q_i. Observe that since $\{x, b\}$ is an odd pair, then all the induced paths from x to b_1 (resp. b_2) avoiding b_2 (resp. b_1) must be even. However, since $\{x, y\}$ is a strong even pair, then by Lemma 3.2 applied to the cutset $\{b_1, b_2\}$ of type IIa, all these induced paths must be odd. This contradiction completes the proof. $\square$

Lemma 3.7. *Let $Q_i = \{a, b\} \in Q(x, y)$ be a cutset of type IIc that separates a pair of vertices $\{x, y\}$ of G. Suppose that all paths in $P(x, a, b, G)$, $P(x, b, a, G)$, $P(y, a, b, G)$ and $P(y, b, a, G)$ have the same parity. Then $\{x, y\}$ is an even pair if and only if $G - Q_i$ has exactly two components.*

Proof. By Lemma 3.2, if $\{x, y\}$ is an even pair, then $G - Q_i$ has exactly two components. Conversely, suppose that $G - Q_i$ has exactly two components. Since every induced path from x and y to a avoiding b and to b avoiding a have the same parity, we have that an induced odd path from x to y, if does exist, necessarily contains a and b. However, given the neighborhoods of Figure 3, a has at most two non-artificial neighbors a_1 and a_2 in H_i, and (a_1, a_2) is an edge. We have the same for a in H_{i-1} and for b in H_{i-1} and H_i. So, every path from x to y containing a and b must contain two such neighbors, say a_1 and a_2. Then, no such path is induced, since (a_1, a_2) is an edge.

□

With Lemmas 3.5, 3.6 and 3.7, we have the main properties of strong even pairs. We are now ready to prove Lemma 3.8 which, together with Lemma 3.1, completes the neccessary theoretical foundation for the proof of Theorem 1.1.

Lemma 3.8. *An introversive graph G contains an even pair if and only if it contains a pair (x, y) of vertices which satisfies the following conditions, where $Q(x, y) = (Q_1, \ldots, Q_k)$ and $Z(x, y) = (z_1, \ldots, z_k)$:*

a. (x, z_1), (z_k, y) *and* (z_i, z_{i+1}), *for* $i = 1, \ldots, k-1$, *are even pairs or odd pairs; and*

b. *the number of odd pairs is even; and*

c. *If* Q_1 *is of type IIa or IIIa,* (x, z_1) *is an even pair and* $Q \subseteq N(x)$*; and*

d. *If* Q_k *is of type IIa or IIIa,* (z_{k-1}, y) *is an even pair and* $Q \subseteq N(y)$*; and*

e. *Every cutset in* $\{Q_2, \ldots, Q_{k-1}\}$ *is of type IIc and generates exactly two components.*

Moreover, a pair (x, y) satisfying these conditions is always an even pair in an introversive graph.

Proof. Let G be an introversive graph. If G contains an even pair, then G contains a strong even pair $\{x, y\}$ which, by Lemmas 3.5, 3.6 and 3.7, satisfies (c), (d) and (e). We have to prove that $\{x, y\}$ satisfies (a) and (b). Let $i \in \{1, \ldots, k-1\}$. By Lemma 3.2, all induced paths from x and y to a vertex of Q_i avoiding the others have the same parity (the same occurs for Q_{i+1}). In H_i, z_i and z_{i+1} plays the role of x and y, respectively. Notice that, by construction, all induced paths from z_i to Q_i in H_i have the same parity and all induced paths from z_{i+1} to Q_{i+1} in H_i have the same parity. Moreover, since $\{x, y\}$ is an even pair, then all induced paths from z_i to z_{i+1} have the same parity. So, $\{z_i, z_{i+1}\}$ is an even pair or an odd pair. By similar arguments, $\{x, z_1\}$ and $\{z_k, y\}$ is an even pair or an odd pair. With this, we have (a). Since $\{x, y\}$ is a strong even pair, then there exists an induced even path $Lxy = Lxa_0 + La_0a_1 + \ldots + La_{k-1}a_k + La_ky$ from x to y, where $a_i \in Q_i$ and each subpath La_ia_{i+1} is an induced path from a_i to a_{i+1} that avoids every other vertex of Q_i and Q_{i+1}. If $\{z_i, z_{i+1}\}$ is an even pair (resp. odd pair), then La_ia_{i+1} must have even (resp. odd) length. Since Lxy has even length, then the number of odd subpaths La_ia_{i+1} is even. So, the number of odd pairs in $\{x, z_0\}, \{z_{k-1}, y\}$ and $\{z_i, z_{i+1}\}$, $0 < i < k$, is even, proving (b).

Conversely, suppose now that we have a pair $\{x, y\}$ that satisfies (a), (b), (c), (d) and (e). If Q_1 is of type IIa or IIIa, then, by (c), x is adjacent to every vertex of Q_1. So, there is no induced path from x to y with two vertices of Q_1. We have the same for Q_k by (d). Given the allowed neighborhoods of Figure 3, we have that the non-artificial neighbors of each vertex of Q_i in H_i and Q_{i+1} in H_{i+1} must be one vertex or two vertices joined by an edge. In both cases, there is no induced path from x to y with two vertices of Q_i (such a path would not be induced), since, by (e), $G - Q_i$ has exactly two components. So, every induced path from x to y contains exactly one vertex from Q_i, where $i = 1, \ldots, k$. With this, we have by (a) that every induced path from x to y have the same parity. Since the number of odd pairs is even, then there is an induced even path from x to y. So, $\{x, y\}$ is an even pair. □

4 The algorithm

Let G be a planar Berge graph. At first, apply Hsu's algorithm to obtain the decomposition tree T of G. Hsu [16] showed that T can be obtained in $O(n^3)$ time and we can check in the same time if the leaves of T are in classes C, L, S or $\{K_3, K_4\}$.

Lemma 3.1 indicates the cases when it is easy to obtain an even pair. If there is a leaf H of T in class C, Hsu [16] showed that we can find in $O(|V(H)|)$ time the set J of C_4-vertices such that $H - J$ is bipartite. By Theorem 2.1, any pair of two distinct vertices $x, y \notin J$ at distance two is an even pair. If there is a leaf H of T in class L with a degree two vertex u, then $N(u)$ is an even pair by Theorem 2.1. If there is a leaf H of T in class S, then we find an even pair in any diamond of H. If T contains a node whose cutset Q is of type I, IIb, $IIIb$ or IV, then we have an even pair in Q. If one of the cases above occurs, then we return the even pair found and we stop the algorithm.

Now assume that no even pair was found in this first phase. Then G is an introversive graph and we will look for even pairs and odd pairs in the leaves of T.

Let us now introduce Z-vertices to T as already explained. First of all, notice that Z-vertices cannot change the induced paths between non-artificial vertices. We will construct a weighted graph W whose vertices are all vertices of G and all Z-vertices included. We will join an edge between two vertices r and s of W if and only if $\{r, s\}$ is an even pair or an odd pair in a leaf H of T. The weight of the edge (r, s) in W is 0 (resp. 1) if $\{r, s\}$ is an even pair (resp. odd pair) of H. We have to answer if this could be done in $O(n^3)$ time.

By including Z-vertices, some leafs of T may not be of class L anymore. Harary et al. [13] proved that a graph G is a line-graph of a bipartite graph if and only if it does not contain odd holes, diamonds ($K_4 - e$) or claws ($K_{1,3}$) as induced subgraphs. Odd holes and claws do not appear with Z-vertices. However, diamonds can emerge when we include a Z-vertex. It is easy to see that this happen only for cutsets of type IIa.

By Lemmas 3.5 and 3.6, we know that if a cutset $Q = \{a, b\}$ of type IIa separates a strong even pair $\{x, y\}$, then x or y (say x) is in a leaf H with Q and there exists a vertex v in H adjacent to a and b such that $\{v, y\}$ is also an even pair. Consequentely, for each leaf H of T and for each Z-vertex z of a IIa cutset $Q = \{a, b\}$ in H, we check if z is in a diamond by looking for a non-artificial vertex v in H that are neighbor of a and b. If there is such a vertex v, we have that $\{z, v\}$ is an even pair and we put it in W. We can do this in $O(n^2)$ time, since we have at most $O(n)$ Z-vertices.

For each leaf H of T, let $Diam(H)$ be the set of Z-vertices that produces diamonds in H. All important even pairs with vertices in $Diam(H)$ was already obtained, including the case when H is a K_3 or a K_4. So, remove from each leaf H the set $Diam(H)$ and delete all leafs isomorphic to K_3 or K_4. With this, we have that every leaf of T is a line graph of a bipartie graph, not necessarily in class L anymore. Theorem below from Hougardy [15] shows us how we can obtain even pairs and odd pairs in line graphs of bipartite graphs.

Theorem 4.1 (Hougardy [15])**.** *Let $\{x, y\}$ be a pair of vertices in a line-graph G of a bipartite graph B. Then $\{x, y\}$ is an even pair (resp. odd pair) if and only if their corresponding edges x_1x_2 and y_1y_2, respectively, in B are such that x_1 and y_1 are in a same partition (resp. different partitions) of B and in different components of $B - \{x_2, y_2\}$.*

Therefore, given a leaf H of T, a vertex x and a Z-vertex z of H, we can apply Theorem 4.1 to decide if $\{z, x\}$ is an even pair, an odd pair or none of them, by analyzing the connectivity of B minus the two edges corresponding to z and x, which takes $O(|V(H)|)$ time, since B has at most $O(|V(H)|)$ edges. So, doing this for each H, x and z, this procedure runs in $O(n^3)$ time, since we have $O(|V(H)|^2)$ such pairs $\{x, z\}$ in H.

With this, we have constructed the weighted graph W. From Lemma 3.8, if there exists two non-artificial vertices $\{x, y\}$ such that there is a path with even length between them in W, then $\{x, y\}$ is an even pair of G. If this does not happen, then G has no strong even pair and, consequently, G has no even pair. We can do this using, for example, Floyd's algorithm that runs in $O(n^3)$ time.

Proof of the Main Theorem 1.1. If G is complete, then G has no even pairs. If G satisfies Lemma 3.1, then we find an even pair of G in $O(n^3)$ time, as explained. If G satisfies Lemma 3.8, then, if G contains an even pair, we can find it in $O(n^3)$ time, as explained. If G has no even pair, then we can also check this in $O(n^3)$ time, as explained before. $\square$

5 Technical proofs

As usual, by w.l.g, we mean *without loss of generality.*

5.1 Proof of Lemma 3.2

Proof of Lemma 3.2. Since $\{x, y\}$ is an even pair, it is easy to see that all paths in $P(x, a, b, H_X)$ have the same parity (if there is one even path and one odd path, we would have an induced odd $\{x, y\}$-path through any path in $P(y, a, b, H_Y)$). Similarly, all paths in $P(x, b, a, H_X)$ have the same parity. If paths in $P(x, a, b, H_X)$ have even length (resp. odd length) and paths in $P(x, b, a, H_X)$ have odd length (resp. even length), then $\{x, a\}$ (resp. $\{x, b\}$) is an even pair. This is a contradiction, since H_X has no even pair. So, all paths in $P(x, a, b, H_X)$ and $P(x, b, a, H_X)$ have the same parity.

Suppose that Q is of type IIc and $H - Q$ has a third component H_Z. So there is an induced odd $\{x, y\}$-path obtained by three subpaths: any induced $\{x, a\}$-path in $P(x, a, b, H_X)$, any induced odd $\{a, b\}$-path in H_Z and any induced $\{b, y\}$-path in $P(y, b, a, H_Y)$. This is a contradiction, since $\{x, y\}$ is an even pair.

Suppose that Q is of type IIa and every path in $P(x, a, b, H_X)$ and $P(x, b, a, H_X)$ have even length. Since $\{x, a\}$ is not an even pair, then there is an induced odd $\{x, a\}$-path in H_X with b. Let $L_{XB} + (b, a)$ be such path (L_{XB} is an induced even $\{x, b\}$-path). Since $\{y, b\}$ is not an even pair, then there is an induced odd $\{y, b\}$-path in H_Y with a. Let $L_{YA} + (a, b)$ be such path (L_{YA} is an induced even $\{y, a\}$-path). However, we have a contradiction, since $L_{XB} + (a, b) + L_{YA}$ is an induced odd $\{x, y\}$-path, but $\{x, y\}$ is an even pair. □

5.2 Proof of Lemma 3.3

Proof of Lemma 3.3. At first, suppose that there is no path from x to a avoiding b and c. So, since (b, c) is an edge, then $\{b, c\}$ is a cutset of type IIa separating x from y. From Lemma 3.2, we have that all paths from x to b avoiding c and to c avoiding b have odd length. So, $\{x, a\}$ is an even pair, contradicting the fact that H_X has no even pair. Therefore, $P(x, a, b, c, H_X)$ is not empty and, with same arguments, $P(x, b, a, c, H_X)$, $P(x, c, a, b, H_X)$, $P(y, a, b, c, H_Y)$, $P(y, b, a, c, H_Y)$ and $P(y, c, a, b, H_Y)$ are not empty.

Applying the same arguments of Lemma 3.2, all paths in $P(x, a, b, c, H_X)$ and $P(y, a, b, c, H_Y)$ have the same parity, as well as all paths in $P(x, b, a, c, H_X)$ and $P(y, b, a, c, H_Y)$, and all paths in $P(x, c, a, b, H_X)$ and $P(y, c, a, b, H_Y)$.

W.l.g., we can consider three possibilities for the parities of paths in $P(x, a, b, c, H_X)$, $P(x, b, a, c, H_X)$ and $P(x, c, a, b, H_X)$:

(i) paths in $P(x, a, b, c, H_X)$ are even; and paths in $P(x, b, a, c, H_X)$ and $P(x, c, a, b, H_X)$ are odd. We call this case *even-odd-odd*.

(ii) paths in $P(x, a, b, c, H_X)$ and $P(x, b, a, c, H_X)$ are even; and paths in $P(x, c, a, b, H_X)$ are odd. We call this case *even-even-odd*.

(iii) paths in $P(x, a, b, c, H_X)$, $P(x, b, a, c, H_X)$ and $P(x, c, a, b, H_X)$ are even. We call this case *even-even-even*.

(iv) paths in $P(x, a, b, c, H_X)$, $P(x, b, a, c, H_X)$ and $P(x, c, a, b, H_X)$ are odd. We call this case *odd-odd-odd*.

In case *even-odd-odd*, we have a contradiction, since $\{x, a\}$ and $\{y, a\}$ are even pairs in H_X and H_Y (which do not have even pairs). In case *even-even-odd*, since $\{x, a\}$ and $\{y, b\}$ are not even pairs, then there exists induced odd paths L_{XBA} and L_{BAY} from x to a through b and from y to b through a, respectively. These paths share the edge (b, a). Again, we have a contradiction, since $L_{XBA} + L_{BAY}$ is an induced odd $\{x, y\}$-path.

So, suppose that we are in case *even-even-even*. Recall that $\{x, b\}$ is not an even pair and an induced path from x to b cannot contain a and c together. So there exists an induced odd path from x to b through a (L_{XAB}) or through c (L_{XCB}). W.l.g., assume that we have L_{XCB}. Since $\{y, c\}$ is not an even pair, there exists an induced odd path L_{YAC} from y to c through a, such that $a \notin L_{YBC}$ and $b \notin L_{YAC}$ (an induced odd path L_{YBC} from y to c through b is impossible, since $xL_{XCB}L_{YBC}y$ would be an induced odd path from x to y.

Notice that $\{x, c\}$ is not an even pair and then there exists an induced odd path from x to c through a (L_{XAC}) or through b (L_{XBC}). First assume that we have L_{XBC}. Again, since $\{y, b\}$ is not an even pair, there exists an induced odd path from y to b through a (L_{YAB}) or through c (L_{YCB}). If we have L_{YCB}, then we have a contradiction, since $xL_{XBC}L_{YCB}y$ is an induced odd path from x to y. So, we have L_{YAB}. Since $\{x, a\}$ is not an even pair, there exists an induced odd path from x to a through b (L_{XBA}) or through c (L_{XCA}). Again we have a contradiction, since we obtain an induced odd path from x to y, namely, $xL_{XBA}L_{YAB}y$ or $xL_{XCA}L_{YAC}y$. Therefore, we conclude that L_{XBC} does not exist.

By consequence, we have L_{XAC}. Again, since $\{y, a\}$ is not an even pair, there exists an induced odd path from y to a through b (L_{YBA}) or through c (L_{YCA}). If we have L_{YCA}, then $xL_{XAC}L_{YCA}y$ is an induced odd path from x and y. By consequence, we have L_{YBA}. Finally, since $\{x, a\}$ is not an even pair, then there exists an induced odd path from x to a through b (L_{XBA}) or through c (L_{XCA}). If we have L_{XCA}, then $xL_{XCA}L_{YAC}y$ is an induced odd path from x to y. So, we have L_{XBA}.

Up to now, we have proved the existence of the induced paths L_{XCB}, L_{YAC}, L_{XAC}, L_{YBA} and L_{XBA}. By combining these paths with paths of $P(y, b, a, c, H_Y)$, $P(x, c, a, b, H_X)$, $P(y, c, a, b, H_Y)$, $P(x, a, b, c, H_X)$ and $P(y, a, b, c, H_Y)$, respectively, we could obtain an induced odd path from x to

y. It implies that a must see all paths in $P(x, c, a, b, H_X)$ and $P(y, c, a, b, H_Y)$; that b see all paths in $P(x, a, b, c, H_X)$ and $P(y, a, b, c, H_Y)$; and that c see all paths in $P(y, b, a, c, H_Y)$.

Let P_{YB} be an induced path from y to b avoiding a and c in H_Y. As observed, there is a vertex $u_1 \in P_{YB}$, such that (c, u_1) is an edge. Let P_{YC} be the induced path $P_{YB}[y, u_1] + c$ from y to c avoiding a and b in H_Y. Again, there is a vertex $u_2 \in P_{YC}$, such that (a, u_2) is an edge. Clearly, we also have that $u_2 \in P_{YB}$. Let P_{YA} be the induced path $P_{YC}[y, u_2] + a$ from y to a avoiding b and c in H_Y. Finally, there is a vertex $u_3 \in P_{YA}$, such that (b, u_3) is an edge. Clearly, we also have that $u_3 \in P_{YB}$. But $P_{YA}[y, u_3] + b$ is induced path from b contained in P_{YB}. Since P_{YB} is induced, we have that $u_1 = u_2 = u_3$ (let us call them only by u). So, $\{a, b, c, u\}$ induces a K_4. If y has paths from each vertex in $\{a, b, c, u\}$ avoiding the others, then $\{a, b, c, u, y\}$ would be a K_5 minor, contradicting the graph planarity.

So, $\{a, b, u\}$, $\{a, c, u\}$ or $\{b, c, u\}$ is a cutset of type IIIa separating y from c, b and a, respectively. So, w.l.g, suppose that $\{a, b, u\}$ is a cutset of type IIIa separating y from c and, consequently, separating y from x. Since we are dealing with the case *even-even-even* for $\{a, b, c\}$, then we have the case *even-even-odd* for $\{a, b, u\}$. This case was also treated and again $\{x, a\}$ or $\{y, b\}$ must be an even pair, contradicting the fact that H_X and H_Y do not have even pairs. This last contradiction shows us that the only possible case is *odd-odd-odd*. □

5.3 Proof of Claim 1

Proof of Claim 1. At first, suppose that Q of type IIIa. In the proof of Lemma 3.3, we showed that, in case *even-even-even*, we have an even pair with x or y and one vertex of Q. Using same arguments, we can prove that, in case *odd-odd-odd*, we have an edge or an odd pair with x or y and one vertex of Q.

Suppose now that Q is of type IIa. In the proof of Lemma 3.2, we proved that, in case *even-even*, $\{x, a\}$ or $\{y, b\}$ (similarly $\{x, b\}$ or $\{y, a\}$) is an even pair. This fact implies that the pairs in one of following four groups is an even pair: $\{x, a\}$ and $\{y, a\}$; $\{x, b\}$ and $\{y, b\}$; $\{x, a\}$ and $\{x, b\}$; or $\{y, a\}$ and $\{y, b\}$. Using same arguments in the case *odd-odd*, we can conclude that at least one pair of the following four groups is an edge or an odd pair: $\{x, a\}$ and $\{y, a\}$; $\{x, b\}$ and $\{y, b\}$; $\{x, a\}$ and $\{x, b\}$; or $\{y, a\}$ and $\{y, b\}$. □

5.4 Proof of Claim 2

Proof of Claim 2. Let x be a non-artificial vertex of H that does not see any vertex of $\{a, b, c\}$. Since H has no even pair, we have by Theorem 2.1 that the neighborhood of x induces a $K_2 \cup K_2$ or a $K_2 \cup K_1$ (see Figure 3). Applying Menger's theorem three times, we have that there exist vertex-disjoint induced paths P_A from x to a avoiding b and c; P_B from x to b avoiding a and c; and P_C from x to c avoiding a and b. Let r, s and t be the neighbors of x such that $r \in P_A$, $s \in P_B$ and $t \in P_C$, respectively. Suppose w.l.g. that (s, t) is an edge, but (r, s) is not. So, P_A or P_B is not just an edge, since (r, s) is not an edge (suppose w.l.g. that P_A is not just an edge).

Now, assume by contradiction that P_A and P_B have the same parity. Therefore, since H has no odd hole, there must be an edge joining a vertex $v \neq x$ of P_A to a vertex w of P_B. Since $v \notin P_B$, then we have by the neighborhood restriction that a neighbor u of v must see w. Notice that $w \neq x$, since P_A is an induced path.

First suppose that $w \neq b$. Then there is two vertices p and q that are neighbors of w in P_B. Since $\{u, v, w\}$ induces a triangle and P_A and P_B are disjoint paths, we have by the neighborhood restriction that $\{w, p, q\}$ must also induce a triangle. Now, the edge (p, q) contradicts the fact that P_B is induced (see Figure 6).

Therefore $w = b$ and we must examine two cases. If $u = x$, then the diamond induced by $\{x, b, v, t\}$ contradicts the required neighborhood for x ($u = x$ is adjacent to $w = b$). If $u \neq x$, then the graph induced by $\{a, b, c, u, v, p\}$ contradicts the required neighborhood for b. □

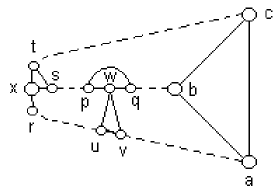

Figure 6: Graph used in the proof of Claim 2

6 Concluding remarks

In this paper, we obtained an $O(n^3)$ time algorithm to determine if a planar Berge graph contains an even pair (and to present one, if it is the case). However, to the best of our knowledge, the problem of determine if a planar non-Berge graph contains an even pair is still open. Bienstock proved that the problem of decide if a general graph contains an even pair is NP-Complete [3]. Unfortunately, this proof uses a non-planar graph in its construction. We summarize the paper result and the corresponding problem left open in the diagram of Figure 7, where the efficient planarity test algorithm is from [14].

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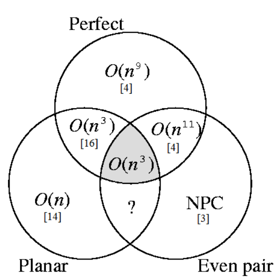

Figure 7: Recognition problems

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