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Spectral properties of KK_n^j graphs

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Abstract

Let G be a simple graph and $A = A(G)$, $L = L(G)$ and $Q = Q(G)$ the adjacency, the laplacian and the signless Laplacian matrices of G, respectively. For each of the associated matrices of G, $M = A, L$, or Q, we call M-spectrum of G the spectrum of the matrix M. In this work we present the KK_n^j graphs, obtained from two copies of the complete graph K_n by adding j edges, $1 \leq j \leq n$, between a vertex of one of the copies and j vertices of the other. We obtain M-spectral properties of this graph based on its clique number and its edge connectivity.

1 Introduction

Let $G = (V, E)$ be a simple graph on n vertices and $D(G) = diag(d_1, \dots, d_n)$ be the diagonal matrix of its vertex degrees. Let $A(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \sim v_j \\ 0, & \text{otherwise.} \end{cases}$$

be the *adjacency* matrix of G. Let $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ be the *Laplacian* and the *signless Laplacian* matrices of G, respectively. For $M(G) = A(G)$, $L(G)$ or $Q(G)$, let $P_M(G, x)$ be the characteristic polynomial of $M(G)$ and $Sp_M(G)$ the spectrum of $M(G)$. A graph G is called M-integral when all eigenvalues of M are integer numbers. Since

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1974, when Harary and Schwenk [6] posed the question *Which graphs have integral spectra?*, the search for A-integral graphs or L-integral graphs has been done. More recently, Q-integral graphs were introduced in the literature [1, 7, 8, 9, 4].
We recall that the clique number $\omega(G)$ of a graph G is the number of vertices in a maximum clique in G. The edge connectivity $\kappa'(G)$ of a graph G is the minimum number of edges whose deletion disconnects G.
In this work we obtain some spectral properties in a special class of graphs, based on these two important parameters.

2 Spectral properties of KK_n^j graphs

Definition 2.1. *Let KK_n^j be the graph obtained from two copies of the complete graph K_n by adding j edges between one vertex of a copy of K_n and j vertices of the other copy, where j is such that $1 \leq j \leq n$.*

Remark: This definition is a generalization of the graph KK_n^2, introduced by Stevanović in [10], where it is analysed other spectral properties of this graph, as Laplacian energy.

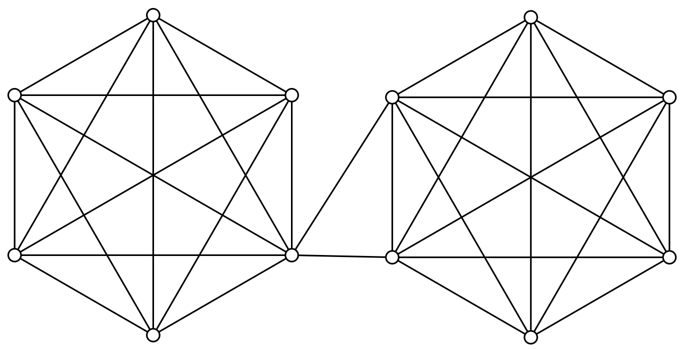

Figure 1: KK_6^2

Remark: We note that $\omega(KK_n^j) = n$ and $\kappa'(KK_n^j) = j$ if $j \leq n-1$ and $\kappa'(KK_n^j) = j-1$ when $j = n$.

The next theorem gives explicitly the expression for the A-characteristic polynomial of KK_n^j:

Theorem 2.2. *If $1 \leq j \leq n$, $n \geq 3$, the A-characteristic polynomial of KK_n^j is $(x+1)^{(2n-4)}h(x)$, where $h(x) = x^4 + (4-2n)x^3 + (n^2 - 6n + 6 - j)x^2 + (2n^2 - 6n + 2nj - j2 - 3j + 4)x + 1 + nj^2 - 2j^2 + n^2 - 2n - 2j + 3jn - jn^2$.*

Proof:

The adjacency matrix of KK_n^j can be represented in the form

$$A = \begin{bmatrix} \mathbb{J}_{n-1} - \mathbb{I}_{n-1} & \mathbf{1}_{n-1} & \mathbf{0}_{n-1,j} & \mathbf{0}_{n-1,n-j} \\ \mathbf{1}_{n-1}^T & 0 & \mathbf{1}_j^T & \mathbf{0}_{n-j}^T \\ \mathbf{0}_{j,n-1} & \mathbf{1}_j & \mathbb{J}_j - \mathbb{I}_j & \mathbb{J}_{j,n-j} \\ \mathbf{0}_{n-j,n-1} & \mathbf{0}_{n-j} & \mathbb{J}_{n-j,j} & \mathbb{J}_{n-j} - \mathbb{I}_{n-j} \end{bmatrix},$$

where $\mathbb{J}$ is the matrix with all entries 1 and $\mathbb{I}$ is the identity matrix.

We will now prove that -1 is an A-eigenvalue with multiplicity at least $2n-4$, exhibiting three kinds of eigenvectors corresponding to it.

We note that, for $\mathbf{u} \in \mathbb{R}^{n-1} \setminus \{0\}$ orthogonal to $\mathbf{1}_{n-1}$ and $\mathbf{v} = \begin{bmatrix} \mathbf{u} \\ \mathbf{0}_{n+1} \end{bmatrix} \in \mathbb{R}^{2n}$ we have that $A\mathbf{v} = -1\mathbf{v}$. Analogously, for $\mathbf{u} \in \mathbb{R}^{n-j} \setminus \{0\}$ orthogonal to $\mathbf{1}_{n-j}$, we have that $\mathbf{w} = \begin{bmatrix} \mathbf{0}_{n+j} \\ \mathbf{u} \end{bmatrix} \in \mathbb{R}^{2n}$ is such that $A\mathbf{w} = -1\mathbf{w}$.

For $\mathbf{u} \in \mathbb{R}^j \setminus \{0\}$ orthogonal to $\mathbf{1}_j$ and $\mathbf{v} = \begin{bmatrix} \mathbf{0}_n \\ \mathbf{u} \\ \mathbf{0}_{n-j} \end{bmatrix} \in \mathbb{R}^{2n}$ we have that $A\mathbf{v} = -1\mathbf{v}$. Again, -1 is an A-eigenvalue of KK_n^j with multiplicity at least $j-1$. Then, -1 is an A-eigenvalue of KK_n^j with multiplicity at least $2n-4$.

We note that the lines on each block of A have constant sum and we can consider now the matrix $\overline{A}$, where the entries are these sums:

$$\overline{A} = \begin{bmatrix} n-2 & 1 & 0 & 0 \\ n-1 & 0 & j & 0 \\ 0 & 1 & j-1 & n-j \\ 0 & 0 & j & n-j-1 \end{bmatrix}.$$

The characteristic polynomial of $\overline{A}$ is $h(x) = x^4+(4-2n)x^3+(n^2-6n+6-j)x^2+(2n^2 - 6n + 2nj - j^2 - 3j + 4)x + 1 + nj^2 - 2j^2 + n^2 - 2n - 2j + 3jn - jn^2$.

It is known that the eigenvalues of $\overline{A}$ are eigenvalues of A, (see Theorem 2.1.3, page 5 in [3]) which completes the proof. ■

The conditions for A-integrality are:

Corollary 2.3. *The graph KK_n^j is A-integral if and only if the roots of $h(x)$ are integers.*

We now obtain the L-characteristic polynomial of KK_n^j:

Theorem 2.4. *If $1 \leq j \leq n$, $n \geq 3$, the L-characteristic polynomial of KK_n^j is $x(x-n)^{2n-j-2}(x-n-1)^{j-1}g(x)$, where $g(x) = x^2 - (n+1+j)x + 2j$.*

Proof: As in the previous theorem, the Laplacian matrix of KK_n^j can be represented in the form of a block matrix

$$L = \begin{bmatrix} (n)\mathbb{I}_{n-1} - \mathbb{J}_{n-1} & -\mathbf{1}_{n-1} & \mathbf{0}_{n-1,j} & \mathbf{0}_{n-1,n-j} \\ -\mathbf{1}_{n-1}^T & n-1+j & -\mathbf{1}_j^T & \mathbf{0}_{n-j}^T \\ \mathbf{0}_{j,n-1} & -\mathbf{1}_j & (n+1)\mathbb{I}_j - \mathbb{J}_j & -\mathbb{J}_{j,n-j} \\ \mathbf{0}_{n-j,n-1} & \mathbf{0}_{n-j} & -\mathbb{J}_{n-j,j} & (n)\mathbb{I}_{n-j} - \mathbb{J}_{n-j} \end{bmatrix}.$$

By a analogous reasoning, if $1 \leq j \leq n$, the L-eigenvalues of KK_n^j are 0, n with multiplicity $2n-j-2$, $n+1$ with multiplicity $j-1$, and $\dfrac{n+j+1 \pm \sqrt{(n+j+1)^2 - 8j}}{2}$ ∎.

In this case we obtain L-integrality conditions directly from the clique number and the edge connectivity of the graph.

Corollary 2.5. *The graph KK_n^j is L-integral if and only if $(n+1+j)^2 - 8j$ is a perfect square.*

Thus we can construct an infinite family of L-integral graphs:

Corollary 2.6. *For all $n \geq 3$, the graph KK_n^n is L-integral.*

In fact, in this case, the roots of $g(x)$ are 1 and $2n$.

Remarks on properties of L-spectrum:

Considering the family $\{KK_n^j : 1 \leq j \leq n\}$, we have that the graph KK_n^n behaves as an extremal graph according to the following aspects:

- The second smallest L-eigenvalue $a(G)$ of a graph G is called the algebraic connectivity of G. Fiedler, in [2] prove that for all G, $a(G) \leq k(G) \leq k'(G)$, where $k(G)$ denotes the vertex connectivity of G.

  As $k(KK_n^j) = 1$ for any j, $1 \leq j \leq n$, $a(KK_n^j) \leq 1$. By Corollary 2.3, $a(KK_n^j)$ attains the maximum value, $a(KK_n^j) = 1$, when $j = n$ and so KK_n^n is an extremal graph with respect to the algebraic connectivity, for all n.

- It is well known that the greatest L-eigenvalue of a graph G, also called the L-index of the graph and denoted by $\mu_1(G)$, is always less than or equal to the size of the graph.

  In our case, for any j, $1 \leq j \leq n$, $\mu_1(KK_n^j) \leq 2n$. Again, based on corollary 2.3, if $j = n$, the L-index attains its maximum value, $\mu_1(KK_n^j) = 2n$ and so KK_n^n is an extremal graph with respect to L-index, for all n.

Finally, we present the Q-characteristic polynomial of KK_n^j and some consequences of it, obtained in [5]

Theorem 2.7. *If $j \leq n \in \mathbb{N}$, $n \geq 3$, the Q-characteristic polynomial of KK_n^j is $(x-2n+2)(x-n+1)^{j-1}(x-n+2)^{2n-j-2}f(x)$, where $f(x) = x^2-(3n+j+3)x+2(n^2+n(j-2)-2j-1)$.*

Consequently, the Q-eigenvalues of KK_n^j are:

- $2n-2$,
- $n-2$ with multiplicity $2n-j-2$,
- $n-1$ with multiplicity $j-1$,
- $\dfrac{3n+j-3 \pm \sqrt{(n-j-1)^2+8j}}{2}$

Corollary 2.8. *The graph KK_n^j is Q-integral if and only if $(n-j-1)^2+8j$ is a perfect square. Moreover, for n, k, $j \in \mathbb{N}$, if one of the conditions bellow is satisfied,*

1. $n = 3j$;
2. $n = 2j-1$;
3. $n = 5k-2$ *and* $j = 3k$;
4. $n = 3k+6$ *and* $j = 2k+6$;
5. $n = j = \dfrac{k(k+1)}{2}$.

then KK_n^j graph is Q-integral:

LQ-integral means L-integral and Q-integral. As an immediate consequence of these results and the characterization of the L-spectrum of the KK_n^j graph, we obtain an infinite family of LQ-integral graphs:

Corollary 2.9. *For all $k \in \mathbb{N}$, if $n = j = \frac{k(k+1)}{2}$, KK_n^j is an LQ-integral graph.*

3 Conclusion

For the three matrices considered, A, L or Q, we obtain the expressions of the characteristic polynomial based on $\omega(KK_n^j)$ and $\kappa'(KK_n^j)$. For $M = L$ or Q we have explicitly the M-spectrum in terms of these two parameters. Finally we built an infinite family of LQ-integral graphs.It remains open the question about the existence of A-integral graphs in the class considered.

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