

## Some problems on idomatic partitions and b-colorings of direct products of complete graphs

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### Abstract

In this note, we deal with the characterization of the idomatic partitions and b-colorings of direct products of complete graphs. We recall some known results on idomatic partitions of direct products of complete graphs and we present new results concerning the b-colorings of the direct product of two complete graphs. Finally, some open problems are given.

## 1 Introduction and preliminary results

Let  $G = (V, E)$  be an undirected finite simple graph without loops (see reference [3] for classical concepts in graph theory). A set  $S \subseteq V$  is called a *dominating set* if for every vertex  $v \in V \setminus S$  there exists a vertex  $u \in S$  such that  $u$  is adjacent to  $v$ . A set  $S \subseteq V$  is called *independent* if no two vertices in  $S$  are adjacent. A set  $S \subseteq V$  is called an *independent dominating set* of  $G$  if it is both independent and dominating. A partition of the vertex set  $V$  into independent dominating sets is called an *idomatic partition* of  $G$  [1, 2]. The maximum size of an idomatic partition of  $G$  is called the *idomatic number*  $id(G)$ . An idomatic partition of a graph  $G$  into  $k$  parts is called an *idomatic  $k$ -partition* of  $G$ . Notice that not every graph has an idomatic  $k$ -partition, for any  $k$ . For example, the cycle graph on five vertices  $C_5$  has

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no idomatic  $k$ -partition for any  $k$ . A *proper coloring* (*coloring* for short) of  $G$  is an assignment of colors to the vertices of  $G$  such that adjacent vertices are assigned different colors. A  $k$ -coloring of  $G$  is a coloring using exactly  $k$  different colors. The smallest number  $k$  such that  $G$  admits a  $k$ -coloring is called the *chromatic number* of  $G$  and is denoted by  $\chi(G)$ . Given a  $k$ -coloring of  $G$ , a vertex  $v$  is said to be *dominant* if  $v$  is adjacent to at least one vertex receiving each of the  $k - 1$  colors not assigned to  $v$ . As remarked by Dunbar et al. in [4], an idomatic partition of a graph  $G$  represents a proper coloring of the vertices of  $G$  where all vertices are dominant. A *b-coloring* of  $G$  [7] is a coloring such that every color class admits at least one dominant vertex. So, b-colorings are relaxed versions of idomatic partitions. Notice that every coloring of  $G$  with  $\chi(G)$  colors is a b-coloring. The *b-chromatic number* of  $G$ , denoted by  $\chi_b(G)$ , is the maximum number  $k$  such that  $G$  admits a b-coloring with  $k$  colors.

The *direct product*  $G \times H$  of two graphs  $G$  and  $H$  is defined by  $V(G \times H) = V(G) \times V(H)$ , and where two vertices  $(u_1, u_2), (v_1, v_2)$  are joined by an edge in  $E(G \times H)$  if  $\{u_1, v_1\} \in E(G)$  and  $\{u_2, v_2\} \in E(H)$ . This product is commutative and associative in a natural way (see reference [6] for a detailed description on product graphs).

Let  $n$  be a positive integer. We denote by  $[n]$  the set  $\{1, \dots, n\}$ . The complete graph  $K_n$  will usually be on the vertex set  $[n]$ .

Idomatic partitions of graphs were studied in [4] as a special coloring problem on graphs defined as *fall colorings*. In that work, the authors show the following result.

**Theorem 1** ([4]). *Let  $n_1 > 1$  and  $n_2 > 1$  be two integers. The direct product graph  $K_{n_1} \times K_{n_2}$  admits an idomatic  $n_1$ -partition and an idomatic  $n_2$ -partition. Furthermore, if  $t > 1$  is an integer such that  $t \notin \{n_1, n_2\}$ , then  $K_{n_1} \times K_{n_2}$  has no idomatic  $t$ -partition.*

Moreover, in [4] it is posed the question of characterizing the idomatic partitions of the direct product of three or more complete graphs. Recently, in [9] it is given a full characterization of the idomatic partitions of the direct

product of three complete graphs. By following the same ideas given in [9], in [8] it is given a characterization of the idomatic sets of a direct product of four complete graphs.

The direct product of graphs  $G_1, G_2, \dots, G_n$  will be denoted  $\times_{i=1}^n G_i$ . Let  $G = \times_{i=1}^k K_{n_i}$  and let  $u = (u_1, \dots, u_k)$  and  $v = (v_1, \dots, v_k)$  be vertices of  $G$ . Then, let

$$e(u, v) = |\{i : u_i = v_i\}|$$

be the number of coordinates in which  $u$  and  $v$  coincide. With this notation we can state that  $u$  and  $v$  are adjacent in  $G = \times_{i=1}^k K_{n_i}$  if and only if  $e(u, v) = 0$ . Therefore,  $I \subseteq V(G)$  is independent if and only if  $e(u, v) > 0$  for any  $u, v \in I$ . Note also that  $e(u, v) \leq k - 1$  holds for any  $u \neq v$ .

Let  $X \subset V(G)$ , where  $G = \times_{i=1}^k K_{n_i}$ , and let

$$\{e(u, v) : u, v \in X \text{ and } u \neq v\} = \{j_1, \dots, j_r\}.$$

Then, we say that  $X$  is a  $T_{j_1, \dots, j_r}$ -set.

This note is organized as follows. Let  $G = \times_{i=1}^k K_{n_i}$ . In Section 2, we summarize the results concerning independent dominating sets and idomatic partitions of  $G$  and we pose two problems. Finally, Section 3 contains some new results concerning  $b$ -colorings of the direct product of two complete graphs and an interesting conjecture is posed.

## 2 Independent dominating sets and idomatic partitions in $\times_{i=1}^k K_{n_i}$

In this section we give two open problems concerning the independent dominating sets and idomatic partitions in the graph  $G = \times_{i=1}^k K_{n_i}$ , with  $k \geq 2$  and  $n_i \geq 2$ . We start by rephrasing known results in terms of  $T_{j_1, \dots, j_r}$ -sets.

Notice first that if  $G = \times_{i=1}^2 K_{n_i}$ , with  $n_i \geq 2$ , and if  $I \subseteq V(G)$  is an independent dominating set, then  $I$  is a  $T_1$ -set (see [4]). For  $k > 2$  we have:

**Proposition 1** ([9]). *Let  $G = \times_{i=1}^3 K_{n_i}$ , with  $n_i \geq 2$ , and let  $I$  be an independent dominating set of  $G$ . Then,  $I$  is either a  $T_1$ -set or a  $T_{1,2}$ -set.*

Moreover, in [9] it is characterized the structure of the independent dominating sets of  $\times_{i=1}^3 K_{n_i}$ . Such results have been extended to the case  $k = 4$  as follows :

**Proposition 2** ([8]). *Let  $G = \times_{i=1}^4 K_{n_i}$ , with  $n_i \geq 2$ , and let  $I$  be an independent dominating set of  $G$ . Then,  $I$  is either a  $T_1$ -set, or a  $T_{1,2}$ -set or a  $T_{1,2,3}$ -set*

**Problem 1.** *Let  $G = \times_{i=1}^k K_{n_i}$ , with  $k, n_i \geq 2$ . Then, for all  $i \in [k - 1]$ , does there exist an independent dominating set of  $G$  which is a  $T_{1,2,\dots,i}$ -set?*

Notice that problem 1 holds for  $k = 2, 3$  and 4 (see [4, 9, 8]). For  $k > 4$  this is an open problem.

Let  $pr_i$  denote the projection homomorphism from  $G$  to the  $i^{th}$  factor  $K_{n_i}$ . It is not difficult to deduce that for each  $i = 1, \dots, k$ , the sets  $pr_i^{-1}(1), \dots, pr_i^{-1}(n_i)$  form an idomatic partition of  $G$  into  $T_{1,2,\dots,k-1}$ -sets and such partitions are the only idomatic partitions of such type (see [9] for details).

For  $k = 3$ , it has been characterized in [9] the idomatic partitions into idomatic  $T_1$ -sets. Moreover, in [9] it is also proved that there exist idomatic partitions composed of  $T_1$ -sets and  $T_{1,2}$ -sets, and it is described how to construct such partitions. Therefore, from the total characterization of the idomatic partitions of the graph  $G = \times_{i=1}^3 K_{n_i}$ , the idomatic number of  $G$  can be easily deduced.

For  $k = 4$ , it has been characterized in [8] the idomatic partitions into idomatic  $T_1$ -sets and an example of idomatic partition into idomatic  $T_{1,2}$ -sets is given. However, it is not known whether there are idomatic partitions formed by various types of idomatic sets.

**Problem 2.** *For  $k \geq 4$ , a complete characterization of the idomatic partitions of the graph  $\times_{i=1}^k K_{n_i}$  is still open.*

### 3 b-colorings of $\times_{i=1}^k K_{n_i}$

In this section we study the b-colorings of the graph  $K_n \times K_m$ , with  $2 \leq n \leq m$ . The main result of this section is the following theorem.

**Theorem 2.** *Let  $G = K_n \times K_m$ , with  $2 \leq n \leq m$ . Let  $\Phi$  be a coloring of  $G$ . Thus,  $\Phi$  is a b-coloring of  $G$  if and only if  $\Phi$  induces an idomatic partition of  $G$ .*

As a consequence of Theorem 2, we have that  $G$  has only two b-colorings corresponding to the only two idomatic partitions of  $G$ , one with  $n$  colors and the other with  $m$  colors. Therefore, the b-chromatic number of  $G$  is equal to  $m$ .

It is clear that any idomatic partition of  $G$  is in fact a b-coloring of  $G$ . In order to prove the converse statement, we will prove the following lemmas.

**Lemma 1.** *Let  $G = K_n \times K_m$  with  $2 \leq n \leq m$ . Assume that  $G$  is b-colored and that vertices  $(i, j)$  and  $(i, t)$  are dominant vertices for different colors  $a$  and  $b$  respectively, where  $i \in [n]$  and  $j, t \in [m]$  with  $j \neq t$ . Then, for any  $k \in [n]$ , with  $k \neq i$ , there exists no dominant vertex  $(k, s)$ , with  $s \in \{j, t\}$ , for a color  $c \notin \{a, b\}$ .*

*Proof.* Let  $a \neq b$  be the colors of vertices  $x = (i, j)$  and  $y = (i, t)$  respectively. By hypothesis,  $x$  and  $y$  are dominant vertices. So, by definition of direct product, there exist two vertices  $x' = (i', j)$  and  $y' = (i'', t)$ , with  $i', i'' \neq i$ , such that vertex  $x'$  is colored with color  $a$  and vertex  $y'$  is colored with color  $b$ . Now, assume that there is a dominant vertex  $z = (k, j)$ , with  $k \neq i, i'$ , colored with a color  $c$  different from  $a$  and  $b$ . Clearly, this is impossible because vertex  $z$  has no neighbor colored with color  $a$ . Therefore,  $z$  can not be a dominant vertex for the color  $c$ . In an analogous way we can deduce that no vertex  $(k, t)$ , with  $k \neq i, i''$ , can be a dominant vertex for a color different from  $a$  and  $b$ .  $\square$

**Lemma 2.** *Let  $G = K_n \times K_m$  with  $2 \leq n \leq m$ . Assume that  $G$  is b-colored and that vertices  $(i, j)$  and  $(i, t)$  are dominant vertices for different colors  $a$*

and  $b$  respectively, where  $i \in [n]$  and  $j, t \in [m]$ , with  $j \neq t$ . For any  $k \notin \{j, t\}$  and  $i', i'' \in [n] \setminus \{i\}$ , with  $i' \neq i''$ , there exist no dominant vertices  $(i', k)$  and  $(i'', k)$  for different colors  $c$  and  $d$  respectively, with  $c$  and  $d$  different from  $a$  and  $b$ .

*Proof.* By Lemma 1, we have that  $k \neq j, t$  and  $i', i'' \neq i$ . Without loss of generality, assume that  $i < i' < i''$ . As in the proof of Lemma 1, we can deduce that there are vertices  $(s_1, j), (s_2, t), (i', p_1)$  and  $(i'', p_2)$  colored with colors  $a, b, c$  and  $d$  resp. with  $s_1, s_2 \neq i$  and  $p_1, p_2 \neq k$ . Now, consider vertex  $(i, k)$ . Such a vertex can not be colored with any color in  $\{a, b, c, d\}$ . Thus, let  $e$  be the color of  $(i, k)$ . Now, as  $(i, j)$  is a dominant vertex for color  $a$ , it must have a neighbor  $(u, v)$  colored with color  $e$ . By construction,  $u \neq i$  and  $v = k$ . Moreover, as  $(i', k)$  is a dominant vertex for color  $c$ , it must have a neighbor  $(u', v')$  colored with color  $e$ , with  $v' \neq k$ , which is not possible. An analogous contradiction is obtained for vertices  $(i, j), (i, t)$  and  $(i'', t)$ .  $\square$

By the commutativity of the direct product, a direct consequence of the previous lemma is the following corollary.

**Corollary 1.** *Let  $G = K_n \times K_m$ , with  $2 \leq n \leq m$ . Then, the  $b$ -chromatic number of  $G$  is equal to  $m$ .*

**Lemma 3.** *Let  $G = K_n \times K_m$ , with  $2 \leq n \leq m$ . Then,  $G$  has only  $b$ -colorings with  $n$  and  $m$  colors. Moreover, such  $b$ -colorings are always idomatic partitions of  $G$ .*

*Proof.* Assume that  $G$  has a  $b$ -coloring with  $k$  colors. By Corollary 1, we know that  $k \leq m$ . Moreover, it is well known that the chromatic number of  $G$  is equal to  $\min\{n, m\} = n$ , and thus  $k \geq n$ . So, assume first that  $n < k < m$ . Let  $s$  be the minimum number of columns containing at least one dominant vertex for each one of the  $k$  color classes. Reordering the columns representing  $G$ , we can assume w.l.o.g. that these are the first  $s$  columns. Moreover, by Lemmas 1 and 2, we can reorder the rows in such a way that the first  $p$  ones contain at least one dominant vertex for each one

of the  $k$  colors classes, where  $p$  is the smallest positive integer for which this property holds. Moreover, we can assume that the first  $t$  rows of these  $p$  ones contain at least two dominant vertices of different colors. We consider the following cases :

- *Case  $t = 0$ .* In this case, by applying Lemma 1 to the columns of  $G$ , we have that  $k$  is at most equal to  $n$ , which is a contradiction to the definition of  $k$ .
- *Case  $t > 0$ .* By Lemmas 1 and 2, we can assume that dominant vertices for different colors in row  $i$  occur in consecutive positions, with  $i \in \{1, \dots, t\}$ . Suppose that  $t = 2$  and assume w.l.o.g. that the dominant vertices for different colors in rows 1 and 2 are  $(1, 1), \dots, (1, u_0)$  and  $(2, u_0 + 1), \dots, (2, u_0 + u_1)$ , with  $u_0, u_1 > 1$  respectively. Now, by using arguments as in the proof of Lemma 1, we know that for each vertex  $(1, h)$ , with  $1 \leq h \leq u_0$ , there exists at least one vertex  $(r, h)$ , with  $r \neq 1$ , having the same color as  $(1, h)$ . Therefore, the color of vertex  $(1, u_0 + w)$ , with  $1 \leq w \leq u_1$ , must be equal to the color assigned to vertex  $(2, u_0 + w)$ . Otherwise, there is a contradiction to the assumption that dominant vertices in row 1 occur in consecutive positions. Moreover, each vertex  $(1, u_0 + w)$  is a dominant vertex as  $(2, u_0 + w)$  which is a contradiction to the minimality of  $p$ . Therefore, we have  $t = 1$ . By repeating the previous reasoning, we can deduce that  $p$  must be equal to 1. Now, we claim that  $s = m$ . Suppose that  $s < m$  and consider vertex  $(1, s + 1)$ . By construction, such a vertex has at least one neighbor colored with each color in  $\{1, \dots, k\}$ , and so, it must be assigned a color not in  $\{1, \dots, k\}$ , which is a contradiction to the definition of  $k$ . So, the only possibility that remains is that  $s = m$ , and so in all cases there is a contradiction to the definition of  $k$ . Therefore,  $k$  must be equal to  $n$  or  $m$ . Finally, note that each vertex  $(1, i)$  is a dominant vertex for the color class  $i$ , for  $i = 1, \dots, m$ . So, each column  $i$  is colored with color  $i$ , which is an idomatic partition of  $G$  into  $m$  idomatic sets.

The same arguments can be used for the rows in order to prove that any

b-coloring of  $G$  with  $n$  colors is an idomatic partition of  $G$  into  $n$  idomatic sets, where each row  $i$  is colored with color  $i$ .  $\square$

We present the main open question of this section as a conjecture.

**Conjecture 1.** *Let  $G = \times_{i=1}^k K_{n_i}$ , with  $k, n_i \geq 2$ . Then, any b-coloring of  $G$  is an idomatic partition of  $G$ .*

## References

- [1] E. J. Cockayne and S. T. Hedetniemi. Disjoint independent dominating sets in graphs. *Discrete Math.*, 15:213–222, 1976.
- [2] E. J. Cockayne and S. T. Hedetniemi. Towards a theory of domination in graphs. *Networks*, 7:247–261, 1977.
- [3] R. Diestel *Graph Theory*. Graduate text in mathematics: 173, Springer-Verlag, 2010.
- [4] J. E. Dunbar, S. M. Hedetniemi, S. T. Hedetniemi, D. P. Jacobs, J. Knisely, R. C. Laskar and D. F. Rall. Fall colorings of graphs. *J. of Comb. Math. and Comb. Comput.*, 33:257–273, 2000.
- [5] T. W. Haynes, S. T. Hedetniemi and P. J. Slater. *Fundamentals of Domination in Graphs*. Marcel Dekker, New York, 1998.
- [6] W. Imrich and S. Klavžar. *Product Graphs: Structure and Recognition*. Wiley-Interscience Series in Discrete Mathematics and Optimization, 2000.
- [7] R. W. Irving and D. F. Manlove. The b-chromatic number of a graph. *Discrete Appl. Math.*, 91:127–141, 1999.
- [8] S. Klavžar and G. Mekiš. Independent dominating sets and idomatic partitions in direct products of four complete graphs. Manuscript, 2010.
- [9] M. Valencia-Pabon. Idomatic partitions of direct products of complete graphs. *Discrete Math.*, 310(5):1118–1122, 2010.

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