THE (p,q)-HELLY PROPERTY AND ITS APPLICATION TO THE FAMILY OF CLIQUES OF A GRAPH

Mitre C. Dourado* **Fábio Protti†**
Jayme L. Szwarcfiter‡

Abstract

Let $p \geq 1$ and $q \geq 0$ be integers. A family $\mathcal{S}$ of sets is (p,q)-*intersecting* when every subfamily $\mathcal{S}' \subseteq \mathcal{S}$ formed by p or less members has total intersection of cardinality at least q. A family $\mathscr{F}$ of sets is (p,q)-*Helly* when every (p,q)-intersecting subfamily $\mathscr{F}' \subseteq \mathscr{F}$ has total intersection of cardinality at least q. A graph G is a (p,q)-*clique-Helly graph* when its family of (maximal) cliques is (p,q)-Helly. According to this terminology, the usual Helly property and the clique-Helly graphs correspond to the case $p = 2, q = 1$. In this work we present characterizations for (p,q)-clique-Helly graphs. For fixed p, q, this characterization leads to a polynomial-time recognition algorithm. When p or q is not fixed, it is shown that the recognition of (p,q)-clique-Helly graphs is NP-hard.

1 Introduction

We say that a family $\mathscr{F}$ of sets *has the Helly property* (or *is Helly*) when every subfamily $\mathscr{F}' \subseteq \mathscr{F}$ of pairwise intersecting sets has non-empty total intersection. When the family of cliques of a graph G satisfies the Helly property, we say that G is a *clique-Helly* graph (cfr. [6]).

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We may think of a more general "p-Helly property", which holds when every $\mathscr{F}' \subseteq \mathscr{F}$ of p-wise intersecting sets has non-empty total intersection.

The p-Helly property has been studied in the context of hypergraphs [1, 2]. In this work we propose a new direction in which the p-Helly property can be generalized, by requiring that the subfamilies $\mathscr{F}' \subseteq \mathscr{F}$ satisfy the following property:

"if every collection of p members of $\mathscr{F}'$ have q elements in common, then $\mathscr{F}'$ has total intersection of cardinality at least q."

This leads naturally to the formal definition of the (p, q)-*Helly property.* According to this terminology, the usual Helly property corresponds to the case $p = 2, q = 1$.

In Section 2, we give a characterization for (p, q)-Helly families of sets. For fixed integers p and q, this characterization leads to a recognition algorithm whose time complexity is polynomial on the size of the family. Still in this section, we consider a slightly generalized form of this property, called the (p, q, r)-*Helly property.* A family $\mathscr{F}$ is said to be (p, q, r)-*Helly* when, for every $\mathscr{F}' \subseteq \mathscr{F}$, if every collection of p members of $\mathscr{F}'$ have q elements in common, then $\mathscr{F}'$ has total intersection of cardinality at least r. We describe a characterization of (p, q, r)-Helly families in terms of the (p, q)-Helly property.

In Section 3, we study the (p, q)-Helly property applied to the case of the family of cliques of a graph. We say that a graph G is (p, q)-*clique-Helly* when its family of cliques is (p, q)-Helly. Clique-Helly graphs are exactly the $(2, 1)$-clique-Helly graphs. We show some examples and properties of (p, q)-clique-Helly graphs and give a characterization for them that leads to a polynomial recognition algorithm for fixed p and q, as we remark in Section 4. We also show in Section 4 that, when p or q is not fixed, recognizing (p, q)-clique-Helly graphs is NP-hard.

Finally, in Section 5 we propose some questions concerning the (p, q, r)-Helly property.

The proofs of the lemmas and theorems of this extended abstract can be found in [4]

1.1 Some definitions and notation

Let G be a graph. A vertex $w \in V(G)$ is a *universal vertex* when w is adjacent to every other vertex of G. If $S \subseteq V(G)$, then we denote by $G[S]$ the subgraph of G induced by S. A subgraph H of G is a *spanning subgraph* of G when $V(H) = V(G)$. A *complete* is a subset of pairwise adjacent vertices. A *clique* is a maximal complete.

If S is a set, then $|S|$ denotes the cardinality of S.

The *universe* $\mathrm{Univ}(\mathscr{F})$ of a family $\mathscr{F}$ of sets is defined as the union of its members: $\mathrm{Univ}(\mathscr{F}) = \cup_{S\in\mathscr{F}} S$. The *total intersection* $\mathrm{Int}(\mathscr{F})$ of a family $\mathscr{F}$ of sets is defined as $\mathrm{Int}(\mathscr{F}) = \cap_{S\in\mathscr{F}} S$. A *core* of a family $\mathscr{F}$ of sets is any subset contained in $\mathrm{Int}(\mathscr{F})$.

We say that S is a *q-set* when $|S| = q$, a *q^--set* when $|S| \leq q$, and a *q^+-set* when $|S| \geq q$. This notation will also be applied to other terms used throughout this work: families, cores, completes and cliques.

2 The Generalized Helly Property

2.1 (p,q)-Helly families of sets

Definition 1 *Let $p \geq 1$ and $q \geq 0$ be integers, and let $\mathscr{F}$ be a family of sets. We say that $\mathscr{F}$ is (p,q)-intersecting when every p^--subfamily $\mathscr{F}' \subseteq \mathscr{F}$ has a q^+-core.*

The following proposition lists some immediate consequences of the above definition:

Proposition 2

(i) *For all $p \geq 1$ and $\mathscr{F}$, $\mathscr{F}$ is $(p,0)$-intersecting.*
(ii) *For all $p > 1$, if $\mathscr{F}$ is (p,q)-intersecting then $\mathscr{F}$ is $(p-1,q)$-intersecting.*
(iii) *For all $q > 0$, if $\mathscr{F}$ is (p,q)-intersecting then $\mathscr{F}$ is $(p,q-1)$-intersecting.*

We remark that, for itens (ii) and (iii) above, the converse is not true in general.

Definition 3 *Let $p \geq 1$ and $q \geq 0$ be integers, and let $\mathscr{F}$ be a family of sets. We say that $\mathscr{F}$ satisfies the* (p,q)-Helly property *when every (p,q)-intersecting subfamily $\mathscr{F}' \subseteq \mathscr{F}$ has a q^+-core. In this case, we also say that $\mathscr{F}$ is (p,q)-*Helly.

The next proposition is also easy to proof:

Proposition 4

(i) *For all* $p \geq 1$ *and* $\mathscr{F}$, $\mathscr{F}$ *is* $(p, 0)$*-Helly.*
(ii) *For all* $p > 1$, *if* $\mathscr{F}$ *is* $(p-1, q)$*-Helly then* $\mathscr{F}$ *is* (p, q)*-Helly.*

The following lemma will be useful for the characterization of (p, q)-Helly families of sets.

Lemma 5 *Let* $p \geq 1$ *and* $q \geq 0$ *be integers,* $\mathcal{Q}$ *a* $(p+1)$*-family of* q*-subsets of* U, *and* $\mathscr{F}$ *a* p^-*-family of sets over* U *such that every member of* $\mathscr{F}$ *contains at least* p *members of* $\mathcal{Q}$. *Then* $\mathscr{F}$ *has a* q^+*-core.*

The case $q = 1$ in the above lemma has been proved in the context of hypergraphs [1].

Since any family of q^+-sets is $(1, q)$-intersecting, it is easy to see that a family $\mathscr{F}$ is $(1, q)$-Helly if and only if the subfamily formed by the q^+-sets of $\mathscr{F}$ has a q^+-core.

Now let us deal with the case $p > 1$. The following theorem presents a characteriztion for (p, q)-Helly families of sets in such a case:

Theorem 6 *Let* $p > 1$ *and* $q \geq 0$ *be integers, and let* $\mathscr{F}$ *be a family of sets. Then* $\mathscr{F}$ *is* (p, q)*-Helly if and only if for every* $(p+1)$*-family* $\mathcal{Q}$ *of* q*-subsets of* $Univ(\mathscr{F})$, *the subfamily* $\mathscr{F}'$ *formed by the members of* $\mathscr{F}$ *that contain at least* p *members of* $\mathcal{Q}$ *has a* q^+*-core.*

By setting $q = 1$, we obtain as a corollary of the above theorem the characterization of k-Helly hypergraphs described in [2].

If $|\text{Univ}(\mathscr{F})| = n$, then the number of $(p+1)$-families of q-subsets of $\text{Univ}(\mathscr{F})$ is $O(n^{q(p+1)})$. Hence, for fixed integers $p > 1$ and $q > 0$, Theorem 6 implies that deciding whether $\mathscr{F}$ is (p, q)-Helly can be done in polynomial time on the size of $\mathscr{F}$.

2.2 (p, q, r)-Helly families of sets

Definition 7 *Let* $p \geq 1$, $q \geq 0$, $r \geq 0$ *be integers, and let* $\mathscr{F}$ *be a family of sets. We say that* $\mathscr{F}$ *satisfies the* (p, q, r)-Helly property *when every* (p, q)*-intersecting subfamily* $\mathscr{F}' \subseteq \mathscr{F}$ *has an* r^+*-core. In this case, we also say that* $\mathscr{F}$ *is* (p, q, r)-Helly.

The above definition has some direct consequences, listed below:

Proposition 8

(i) *For all $p \geq 1$ and $q \geq 0$, $\mathscr{F}$ is (p,q)-Helly if and only if $\mathscr{F}$ is (p,q,q)-Helly.*
(ii) *For all $p \geq 1$, $q \geq 0$ and $\mathscr{F}$, $\mathscr{F}$ is $(p,q,0)$-Helly.*
(iii) *For all $p > 1$, if $\mathscr{F}$ is $(p-1,q,r)$-Helly then $\mathscr{F}$ is (p,q,r)-Helly.*
(iv) *For all $q > 0$, if $\mathscr{F}$ is $(p,q-1,r)$-Helly then $\mathscr{F}$ is (p,q,r)-Helly.*
(v) *For all $r > 0$, if $\mathscr{F}$ is (p,q,r)-Helly then $\mathscr{F}$ is $(p,q,r-1)$-Helly.*
(vi) *For all $q,r \geq 0$, $\mathscr{F}$ is $(1,q,r)$-Helly if and only if the subfamily formed by the q^+-sets of $\mathscr{F}$ has an r^+-core.*
(vii) *For all $r \geq q \geq 0$, $\mathscr{F}$ is (p,q,r)-Helly if and only if $\mathscr{F}$ is (p,r,r)-Helly.*

Because of the item (vii) above, from now on we assume that $q \geq r$.

We describe now a characterization of (p,q,r)-Helly families of sets in terms of the (p,q)-Helly property.

Let $p \geq 1$ and $q \geq r \geq 0$ be integers, and let $\mathscr{F}$ be a family of sets. Denote by $X = \{X_1, \ldots, X_{|X|}\}$ the collection of the (p,r)-intersecting subfamilies of $\mathscr{F}$ which are *not* (p,q)-intersecting. Let $I = \{1,2,\ldots,|X|\}$. For each $F_j \in \mathscr{F}$, denote $I(F_j) = \{i \in I \mid F_j \in X_i\}$. For $i,k \in I$, represent by R_i an r-set formed by chosen elements that satisfy $R_i \cap R_k = \emptyset$ for $i \neq k$ and $R_i \cap \text{Univ}(\mathscr{F}) = \emptyset$. The *augmentation of $\mathscr{F}$ relative to (q,r)* is a family $\mathscr{A}$ of sets, obtained from $\mathscr{F}$, as follows. For each $\mathscr{F}_j \in \mathscr{F}$, the corresponding member of $\mathscr{A}$ is $A_j = \mathscr{F}_j \cup (\bigcup_{i \in I(F_j)} R_i)$.

Theorem 9 *Let $p \geq 1$ and $q \geq r \geq 0$ be integers. A family $\mathscr{F}$ of sets is (p,q,r)-Helly if and only if the augmentation of $\mathscr{F}$ relative to (q,r) is (p,r)-Helly.*

3 (p,q)-clique-Helly Graphs

3.1 Definition and Examples

We start this section by applying the concepts of the previous section to the family of cliques of a graph:

Definition 10 *Let $p \geq 1$ and $q \geq 0$ be integers, and let G be a graph. We say that G is a* (p,q)-clique-Helly graph *when its family of cliques is (p,q)-Helly.*

In the remainder of this work, we will assume that $p \geq 2$ and $q \geq 1$, unless differently mentioned.

It is clear that $(p-1,q)$-clique-Helly graphs form a subclass of (p,q)-clique-Helly graphs. The example below shows other relations between classes of (p,q)-clique-Helly graphs:

Example 11 Define the graph $G_{p,q}$ in the following way: $V(G_{p,q})$ is formed by a $(q-1)$-complete Q, a p-complete $Z = \{z_1, \ldots, z_p\}$, and a p-independent set $W = \{w_1, \ldots, w_p\}$. Moreover, there exist the edges (z_i, w_j), for $i \neq j$, and the edges (q, x), for $q \in Q$ and $x \in Z \cup W$. Figure 1 depicts a scheme of the graph $G_{p,q}$, where a dashed line between z_i and w_i means $(z_i, w_i) \notin E(G_{p,q})$.

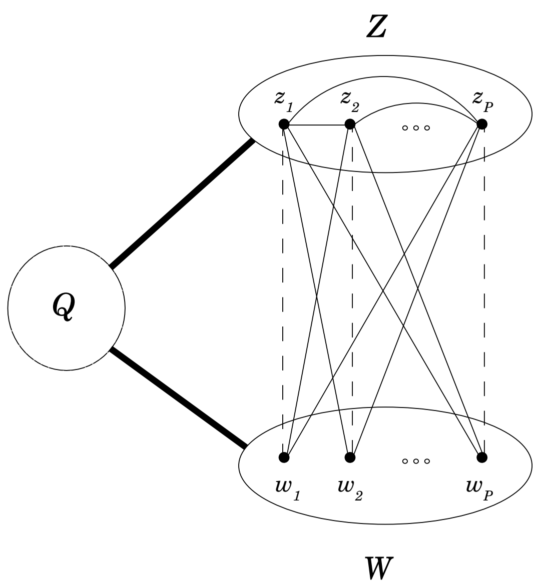

Figure 1: The graph $G_{p,q}$.

The family of cliques of the graph $G_{p,q}$ contains exactly $p+1$ members, each one of size $p+q-1$: $Q \cup \{z_1, \ldots, z_p\}$ and $Q \cup (Z \backslash \{z_i\}) \cup \{w_i\}$, for $1 \leq i \leq p$.

Observe that $G_{p,q}$ is (p,q)-clique-Helly, but it is not $(p-1,q)$-clique-Helly. Therefore, $G_{p,q}$ is (t,q)-clique-Helly for $t \geq p$, and it is not (t,q)-clique-Helly for $t < p$.

Moreover, $G_{p+1,q}$ is not (p,q)-clique-Helly, but it is (p,t)-clique-Helly for any $t \neq q$. Consequently, for distinct q and t, (p,q)-clique-Helly graphs and (p,t)-clique-Helly graphs are incomparable classes.

Define a graph G to be K_r-free when the size of the maximum clique of G is at most $r-1$. An interesting fact derived from Definition 10 is that every $K_{(p+q)}$-free graph is (p_1,q_1)-clique-Helly for $p_1 \geq p$ and $q_1 \geq q$.

Theorem 12 *Let G be a $K_{(p+q)}$-free graph. Then G is (p_1,q_1)-clique-Helly for all $p_1 \geq p$ and $q_1 \geq q$.*

3.2 Characterizing (p,q)-clique-Helly Graphs

In order to give a characterization for (p,q)-clique-Helly graphs, we need some further definitions and lemmas, presented in the sequel.

Definition 13 [8] *Let $\mathscr{F}$ be a subfamily of cliques of G. The* clique subgraph induced by $\mathscr{F}$ in G, *denoted by $G[\mathscr{F}]_c$, is the subgraph of G formed exactly by the vertices and edges belonging to the cliques of $\mathscr{F}$.*

Definition 14 *Let G be a graph, and let C be a p-complete of G. The* p-expansion relative to C *is the subgraph of G induced by the vertices w such that w is adjacent to at least $p-1$ vertices of C.*

We remark that the p-expansion for $p=2$ has been used for characterizing clique-Helly graphs [5, 8]. It is clear that constructing a p-expansion relative to a given p-complete C can be done in polynomial time, for a fixed p.

Lemma 15 *Let G be a graph, C a p-complete of it, H the p-expansion of G relative to C, and $\mathscr{C}$ the subfamily of cliques of G that contain at least $p-1$ vertices of C. Then $G[\mathscr{C}]_c$ is a spanning subgraph of H.*

Definition 16 *Let G be a graph. The graph $\Phi_q(G)$ is defined in the following way: the vertices of $\Phi_q(G)$ correspond to the q-completes of G, two vertices being adjacent in $\Phi_q(G)$ if the corresponding q-completes in G are contained in a common clique.*

Observe that $\Phi_q(G)$ can be constructed in polynomial time, for a fixed q. We also remark that Φ_q is precisely the operator $\Phi_{q,2q}$, studied in [7]. An interesting property of Φ_q is that it preserves the subfamily of cliques of G containing at least q vertices:

Lemma 17 (Clique Preservation Property) *Let G be a graph. Then there exists a bijection between the subfamily of q^+-cliques of G and the family of cliques of $\Phi_q(G)$.*

The graph $\Phi_2(G)$ is the *edge clique graph* of G, introduced in [3], where the validity of the Clique Preservation Property was shown to that case.

The following definition is possible due to the Clique Preservation Property:

Definition 18 *Let G be a graph. If C is a q^+-clique of G, denote by $\Phi_q(C)$ the clique that corresponds to C in $\Phi_q(G)$. If C' is a clique of $\Phi_q(G)$, denote by $\Phi_q^{-1}(C')$ the q^+-clique that corresponds to C' in G. If $\mathscr{F}$ is a subfamily of q^+-cliques of G, define $\Phi_q(\mathscr{F}) = \{\Phi_q(C) \mid C \in \mathscr{F}\}$. If $\mathscr{C}$ is a subfamily of cliques of $\Phi_q(G)$, define $\Phi_q^{-1}(\mathscr{C}) = \{\Phi_q^{-1}(C) \mid C \in \mathscr{C}\}$.*

Lemma 19 *Let G be a graph, $\mathscr{F}$ a subfamily of q^+-cliques of it, $\mathscr{C} = \Phi_q(\mathscr{F})$, and $H = \Phi_q(G)$. Then $H[\mathscr{C}]_c$ contains a universal vertex if and only if $G[\mathscr{F}]_c$ contains q universal vertices.*

Lemma 20 *Let C be a $(p+1)$-complete of a graph G, and let $\mathscr{C}$ be a p^--subfamily of cliques of G such that every clique of $\mathscr{C}$ contains at least p vertices of C. Then $\mathscr{C}$ has a 1^+-core.*

Now we are able to present a characterization for (p, q)-clique-Helly graphs. The cases $p = 1$ and $p > 1$ will be dealt with separately, as in Section 2.

Theorem 21 *Let G be a graph, and let W be the union of the q^+-cliques of G. Then G is a $(1, q)$-clique-Helly graph if and only if $G[W]$ contains q universal vertices.*

Theorem 22 *Let $p > 1$ be an integer. A graph G is a (p, q)-clique-Helly graph if and only if every $(p+1)$-expansion of $\Phi_q(G)$ contains a universal vertex.*

4 Complexity Aspects

Let p and q be fixed positive integers. If $p = 1$, testing whether the union of the q^+-cliques of G contains q universal vertices (Theorem 21) can be easily done in polynomial time. If $p > 1$, testing the existence of a universal vertex in every $(p+1)$-expansion of $\Phi_q(G)$ (Theorem 22) can also be done in polynomial time, since the number of such $(p+1)$-expansions is $O(|V(G)|^{q(p+1)})$. Thus:

Corollary 23 *For fixed positive integers p, q, there exists a polynomial time algorithm for recognizing (p, q)-clique-Helly graphs.*

But when p (or q) is not fixed, the problem of deciding whether a given graph G is (p, q)-clique-Helly is NP-hard.

Theorem 24 *The problem of recognizing (p, q)-clique-Helly graphs when p (or q) is part of the input of the problem is NP-hard.*

5 Some Questions

It remains open the question of deciding whether there exists a recognition algorithm for (p, q, r)-families of sets which is polynomial on the size of the input family, for fixed integers p, q and r.

Define a graph to be *(p, q, r)-clique-Helly* if its family of cliques is (p, q, r)-Helly. Another interesting question is to obtain a characterization for (p, q, r)-clique-Helly graphs that might possibly lead to a polynomial time recognition algorithm on the size of the input graph, for fixed p, q and r.

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Universidade Federal do Rio de Janeiro
COPPE
Caixa Postal 68511
21945-970, Rio de Janeiro, RJ, Brasil
E-mail: mitre@cos.ufrj.br

Universidade Federal do Rio de Janeiro
NCE
Caixa Postal 2324
20001-970, Rio de Janeiro, RJ, Brasil
E-mail: fabiop@nce.ufrj.br

Universidade Federal do Rio de Janeiro
Instituto de Matemática, NCE e COPPE
Caixa Postal 2324
20001-970, Rio de Janeiro, RJ, Brasil
E-mail: jayme@nce.ufrj.br