COMPARABILITY $\cap$ HELLY IS NOT FIXED UNDER K^2

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Abstract

The clique operator maps a graph G into its clique graph $K(G)$; which has as vertices all the cliques of G and two vertices are adjacent if their corresponding cliques have non-empty intersection. As far as we know all Helly graphs subclasses, studied up to now, are fixed or closed under K or fixed under K^2. It is easy to see that $Comparability \cap Helly$ is not closed under K but it is under K^2. We prove that this class is not fixed under K^2 showing a comparability and Helly graph $G^{\#}$ for which there is no comparability and Helly graph G, with $K^2(G) = G^{\#}$. The conclusion follows from results obtained about the Helly property on the family of maximal chains of a poset.

1 Introduction and Basic Definitions

A *graph* is a pair $(V(G), E(G))$ where $V(G)$ and $E(G)$ are the vertex set and edge set of G, respectively. An edge with x and y as extremes is noted by xy or yx. In this note all graphs are simple, i.e., without loops or multiple edges.

A graph G is a *comparability* graph if there is a partial order relation: $\leq$ on $V(G)$ such that $xy \in E(G)$ if and only if $x < y$ or $x > y$, we say that $\leq$ is *associated* with G. By *Comparability* we denote the class of all comparability graphs. A *clique* of a graph G is a maximal complete set of vertices of G.

A family of sets has the *Helly* (resp. *m-Helly*) property if each pairwise intersecting subfamily (resp. with at most m members) has non-empty intersection. We say that a graph whose family of cliques has the Helly property is Helly and we call *Helly* the class of all Helly graphs.

Keyword: clique operator, comparability graphs, Helly property, maximal chains.

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The operator K was studied in different classes of graphs and three behaviours were observed. First, we can mention that K of the class of interval graphs is the class of proper interval graphs [6]. By consequence, this class is closed under K.

Second, K of the class proper interval graphs is the class of proper interval graphs [6], i.e. this class is fixed under K. Later, Bandet and Prisner [1] generalized this result, finding sufficient conditions for a Helly class to be fixed under K.

We can observe the last behaviour through the following example. In [2, 4] the authors proved that K iterates between $Chordal \cap Helly$ and $DuallyChordal$. Moreover these results were generalized in [5] for all Helly classes of intersection graphs of paths in a tree [7].

The characterization of $K(\mathit{Comparability})$ is still open, and concerning with this, it is known that $K(\mathit{Comparability}) \not\subseteq \mathit{Comparability}$ and $K(\mathit{Comparability} \cap \mathit{Helly}) \not\subseteq \mathit{Comparability} \cap \mathit{Helly}$, (see figure 1). Since Helly is a class fixed under K [3], $\mathit{Comparability}$ is an hereditary class and $K^2(G)$ is an induced subgraph of G for all Helly graphs [3], it follows that $K^2(\mathit{Comparability} \cap \mathit{Helly}) \subseteq \mathit{Comparability} \cap \mathit{Helly}$. We wondered if both classes were equal or in other words, if $\mathit{Comparability} \cap \mathit{Helly}$ is fixed under K^2. In this work we conclude that the answer to this question is no.

Escalante [3] proved that if G is a reduced (i.e. with no twins vertices) Helly graph, then $K^2(G)$ is a subgraph of G obtained by deleting dominated vertices of G. This is the reason we studied the dominated vertices in comparability graphs.

Since each comparability graph G has an associated partial order on $V(G)$, $\leq$, and the cliques of G are exactly the maximal chains of $(V(G), \leq)$, we studied the Helly property on the family of maximal chains of a poset.

We say that $(A, \leq)$ is a poset if A is a non-empty set and $\leq$ is a partial order on A.

The article is organized as follows. In section 2 we present the results concerning posets. We prove that the family of maximal chains of a poset has the Helly property if and only if it has the 3-Helly property, and also characterize these posets by a forbidden configuration on their diagrams.

In section 3 we present a graph $G^{\#}$ and applying the results of the previous section we justify that it is not the image under K^2 of any comparability Helly graph.

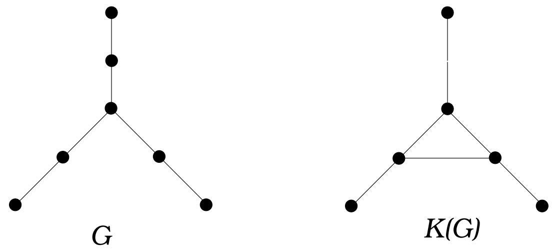

Figure 1: $K(Comparability \cap Helly) \not\subseteq Comparability \cap Helly$

2 Poset's results

Given a poset $(A, \leq)$, an element $x \in A$ is *covered* by $y \in A$ if $x < y$ and there is no $z \in A$ such that $x < z < y$. For an element a of A we define: $\widehat{C_a} = \{y \in A \mid a \text{ is covered by } y\}$ $\check{C}_a = \{y \in A \mid y \text{ is covered by } a\}$. We denote $v \sim a$ if and only if $[v \leq a \vee v \geq a]$.

Definition 1 *Let $(A, \leq)$ be a poset, $y, z \in A$ with $z \leq y$. We say that z is* **above dominated** *by y if*

$$x \geq z \Rightarrow x \sim y$$

Analogously, if $z \geq y$, we say that z is **below dominated** *by y if*

$$x \leq z \Rightarrow x \sim y$$

Observations

1. If z is below dominated by y and $y \in \check{C}_z \Rightarrow \check{C}_z = \{y\}$.

2. If z is below dominated by y_1 and also by y_2, then $y_1 \sim y_2$.

3. Let $(A, \leq)$ be a poset, and $a, b \in A$ such that $a < b$ and b is not below dominated by a. Then there exists $v < b \mid v \not\sim a$.

4. If x is below (or above) dominated by y and M is a maximal chain containing x, then $y \in M$.

Lemma 1 *Let $(A, \leq)$ be a poset, $a, b \in A$, such that $a \not\sim b$ and they are not both below dominated by a same vertex. Then there exist a' and b', not both below dominated by a same vertex, such that $a' = a$ (respectively $b' = b$) whenever a (respectively b) is minimal, else $a' \in \check{C}_a$ (respectively $b' \in \check{C}_b$).*

Proof: If a and b are minimal, then it is obvious. In case that only one of these vertices is minimal, let say b is minimal and $\check{C}_a$ is non-empty. Since a is not below dominated by b, then there exist $a' \in \check{C}_a$ such that a' is not comparable with b and therefore, not below dominated by b. Then let consider the case neither a nor b is minimal. $\check{C}_a = \{a_1 \dots a_n\}$ and $\check{C}_b = \{b_1 \dots b_m\}$. Suppose that each pair $\{a_i, b_j\}$ with $1 \leq i \leq n$ and $1 \leq j \leq m$ is below dominated by a vertex w_{ij}. We can assume that $w_{ij} = min\{w \mid a_i$ and b_j are below dominated by $w\}$. This minimum exists by Observation 2. If $w_{ij} = z$ for every $i = 1 \dots n$ and $j = 1 \dots m$, a and b are both below dominated by z, a contradiction. Since b_1 is below dominated by w_{11} and w_{21}, these vertices are comparable. We claim that $w_{11} = w_{21}$. In fact, suppose that $w_{11} < w_{21}$. By minimality of w_{21}, we know that a_2 is not below dominated by w_{11}. Then, by Observation 3, there is $x < a_2$ such that $x \not\sim w_{11}$. Since a_2 is below dominated by w_{21} then $x \sim w_{21}$. If $x > w_{21}$ then $x > w_{11}$, a contradiction. If $x < w_{21}$ then $x < b_1$ and since b_1 is below dominated by w_{11} then $x \sim w_{11}$, a contradiction. Thus $w_{11} = w_{21}$. Let $w_1 = w_{11} = w_{21} = \dots = w_{n1}$. In general, $w_j = w_{1j} = w_{2j} = \dots = w_{nj}$, observe that w_j is the minimum vertex which below dominates $\{a_1, a_2 \dots a_n, b_j\}$ for $1 \leq j \leq m$. Since a_1 is below dominated by $w_1, w_2 \dots w_m$, this set is totally ordered and we can assume, without loss of generality, that $w_1 \geq w_2 \geq \dots \geq w_m$.

Now we claim $w_i = w_j$ for $i \neq j$. Suppose $w_j < w_i$. By minimality, b_i is not below dominated by w_j. Then there exists $v < b_i \mid v \not\sim w_j$. Since b_i is below dominated by w_i we have $v \sim w_i$. In case $v > w_i \Rightarrow v \sim w_j$, a contradiction. On the other hand, $v < w_i \Rightarrow v < a_1$ and $v \not\sim w_j$. Then a_1 is not below dominated by w_j, a contradiction.

It follows that $w_i = w_j$ for $i \neq j$. Therefore, there is a vertex w which below dominates a and b, a contradiction.

□

Lemma 2 *Let $(A, \leq)$ be a poset, $a, b \in A$ such that $a \leq b$ and they are not both below dominated by a same vertex. Then, there exist $b' \in \{v < b \mid v \not\sim a\}$, such that a and b' are not both below dominated by a same vertex.*

Proof: Since b is not below dominated by a, there exists $v < b$ such that $v \not\sim a$. Let $\{b_j, j : 1 \dots n\}$ be all the vertices which satisfy these conditions. Suppose that each pair $\{a, b_j\}$ with $1 \leq j \leq n$ is below dominated by a vertex z_j. Then the set $\{z_j, 1 \leq j \leq n\}$ is linearly ordered and we can consider

$z = min\{z_j, 1 \leq j \leq n\}$.

Now, let $x < b$. If $x < a$ then $x \sim z$; if $a < x < b$ then $x > z$ and thus $x \sim z$ and in case $x \not\sim a$ then $x = b_k$ with $1 \leq k \leq n$ and b_k is below dominated by z_k. But $z < z_k < b_k$ and that implies $x \sim z$. Therefore, b results below dominated by z, a contradiction.

□

Lemma 3 *Let $(A, \leq)$ be a poset, and $a, b \in A$ such that they are not both below dominated by a same vertex in $(A, \leq)$. Then, there exist two disjoint chains: $m_1 - a$ and $m_2 - b$, where m_1 and m_2 are minimals of A.*

Proof: We have two cases. Suppose first that $a \not\sim b$, by Lemma 1, there are vertices a_1, b_1 with $a_1 = a$ or $a_1 \in \check{C}_a$ and $b_1 = b$ or $b_1 \in \check{C}_b$ such that they are not both below dominated by a same vertex in A. In case $a \leq b$, by Lemma 2, there are a_1, b_1 with $a_1 = a$ and $b_1 \in \{x < b \mid x \not\sim a\}$ such that they are not both below dominated by a same vertex in A. It is clear that the chains: $(a_1 - a)$ (in case $a_1 = a$ the chain $(a_1 - a)$ is simply a) and $(b_1 - b)$ (where $(b_1 - b)$ is any chain from b_1 to b), verify $(a_1 - a) \cap (b_1 - b) = \emptyset$. Iterating this procedure with $a := a_1$ and $b := b_1$ until a and b are minimals of A, the result follows. Indeed, we know that $(a_n - a_{n-1}) \cap (b_n - b_{n-1}) = \emptyset$. If $(a_n - a_{n-1}) \cap (b_m - b_{m-1}) \neq \emptyset$ for any $n > m$, let say, $a_n = b_{m-1}$ then $a_{n-1} \geq b_{m-1}$ and therefore $a_{n-1} \geq b_{n-1}$. But in this case, by Lemma 2, a_n should be not comparable with b_{n-1}, a contradiction.

□

In what follows, we will characterize the posets whose family of maximal chains satisfies the 3-Helly property, by presenting a forbidden configuration on their diagram.

Definition 2 *Let $(A, \leq)$ be a poset. $(A, \leq)$ has the 2GN configuration if its diagram has a cycle C which is the union of two paths: $(\mu < \ldots < d < \ldots < \eta) \cup (\mu < \ldots < \eta)$ such that: d and μ are not both below dominated by a same vertex, d and η are not both above dominated by a same vertex.*

Theorem 1 *Let $(A, \leq)$ be a poset. The family of the maximal chains of $(A, \leq)$ is not 3-Helly if and only if it has a 2-GN configuration.*

Proof: Suppose first that $(A, \leq)$ has a 2-GN configuration as described before. By Lemma 3, since d and μ are not below dominated by a same vertex there exist two disjoint subchains $(m_1 - d)$ and $(m_2 - \mu)$ with m_1 and m_2 minimal of $(A, \leq)$. Analogously, there exist two disjoint subchains $(d - M_1)$ and $(\eta - M_2)$ with M_1 and M_2 maximal of $(A, \leq)$. Let C_1 and C_2 be the two subchains of the cycle C defined as follows: C_1 contains μ, d, η and $C_2 = C - C_1$. We construct three pairwise intersecting chains with empty total intersection as follows: $F_1 = \underbrace{(m_2 - \mu)} \cup \underbrace{(\mu - \eta)}_{\subseteq C_2} \cup \underbrace{(\eta - M_2)}$. $F_2 = \underbrace{(m_1 - d)} \cup \underbrace{(d - \eta)}_{\subseteq C_1} \cup \underbrace{(\eta - M_2)}$. $F_3 = \underbrace{(m_2 - \mu)} \cup \underbrace{(\mu - d)}_{\subseteq C_1} \cup \underbrace{(d - M_1)}$.

Conversely, suppose that the family does not satisfy the $3 - Helly$ property. Then there exist F_1, F_2, F_3, three pairwise intersecting maximal chains with empty total intersection. Let $x \in F_1 \cap F_2$; $y \in F_2 \cap F_3$ and $z \in F_3 \cap F_1$. Without loss of generality we suppose $x < y < z$. Let $x^* = max\{w \in F_1 \cap F_2 : w < y\}$ and $z^* = min\{w \in F_1 \cap F_3 : w > y\}$. Then $\underbrace{(x^* < \ldots < y)}_{\subseteq F_2} \cup \underbrace{(y < \ldots < z^*)}_{\subseteq F_3} \cup \underbrace{(x^* < \ldots < z^*)}_{\subseteq F_1}$ is a cycle and since $F_1 \cap F_2 \cap F_3 = \emptyset$, observation 4 tells us that x^* and y are not below dominated by a same vertex and that y and $z^*$are not above dominated by a same vertex. Thus $(A, \leq)$ has the 2-GN configuration.

□

The next theorem will show that having the 3-Helly property is a sufficient condition to have the Helly property for the family of maximal chains of any poset.

Theorem 2 *Let $\mathcal{F}$ be the family of maximal chains of a poset $(A, \leq)$ and $n \geq 4$. Then $\mathcal{F}$ has the (n-1)-Helly property if and only if it has the n-Helly property.*

Proof: Suppose that $\mathcal{F}$ has the $(n-1)$-Helly property but it does not satisfy the n-Helly property. Then there exists an intersecting subfamily $(C_j)_{j \in J}$, with $|J| = n$, $\bigcap_{j \in J} C_j = \emptyset$ but for every $I \subset J$, we have that $\bigcap_{i \in I} C_i \neq \emptyset$. Let $a_1 \in \bigcap_{i \neq 1} C_i$, $a_1 \notin C_1$; $a_2 \in \bigcap_{i \neq 2} C_i$, $a_2 \notin C_2 \ldots a_n \in \bigcap_{i \neq n} C_i$ and $a_n \notin C_n$. Since $a_1, a_2 \ldots a_n$ are all different and pairwise comparable, we can suppose that $a_1 < a_2 < \ldots < a_n$. Take any j such that $2 < j \leq n-1$. Since $a_j \notin C_j$ and $a_{j-1}, a_{j+1} \in C_j$ we can consider $\mu = max\{x < a_{j-1} \mid x \in C_{j-1} \cap C_j\}$,

$\eta = min\{x > a_{j-1} \mid x \in C_{j-1} \cap C_j\}$ and take the cycle $\underbrace{(\mu - \eta)}_{C_{j-1}} \cup \underbrace{(\mu - a_{j-1} - \eta)}_{C_j}$.

Observe that μ and a_{j-1} are not both below dominated by a same vertex. Indeed, suppose that z below dominates μ and a_{j-1}. Since $a_{j-1} \in \bigcap_{i \neq j-1} C_i$ then $z \in \bigcap_{i \neq j-1} C_i$. On the other hand, since $\mu \in C_{j-1}$ then $z \in C_{j-1}$. Therefore $z \in \bigcap_{1 \leq i \leq n} C_i$, a contradiction. In the same way, η and a_{j-1} are not both above dominated by a same vertex. Then $(A, \leq)$ has a 2-GN configuration. Thus, $\mathcal{F}$ does not satisfy the 3-Helly property, a contradiction. The converse is trivial.

□

3 Comparability Graph's results

We recall some definitions that we use in this section. Let u be a vertex of a graph G. $N[u] = \{u\} \bigcup \{v \in V(G) \mid vu \in E(G)\}$. We say that u and v are equivalent when $N[u] = N[v]$. G^* will denote the graph obtained by cancelling out the equivalence relation, i.e., the vertices are equivalence classes, and adjacency holds between equivalence classes if and only if it holds between their representatives. Finally, if $G \simeq G^*$ we say G is reduced.

A vertex u of a graph G is *dominated* by a different vertex v if its closed neighborhood, $N[u]$, is contained in $N[v]$.

Let us remind the following theorem due to Escalante.

Theorem 3 *[3] Let G be a Helly graph. $K^2(G)$ is the graph obtained deleting dominated vertices of the reduced graph of G.*

In spite of this theorem, if G is a Helly graph and we are interested in finding the set $K^{-2}(G) \cap Helly$ we have to consider:

- If $H \in Helly$ such that $K^2(H) = G$, then G is an induced subgraph of H.
- If H^* is the reduced graph of H, $K^2(H) = K^2(H^*)$.
- A graph is helly if and only if its reduced graph is so.

From the previous observations we can limit ourselves to reduced graphs. If H is a reduced graph of $K^{-2}(G) \cap Helly$, then no vertex in G is dominated in H.

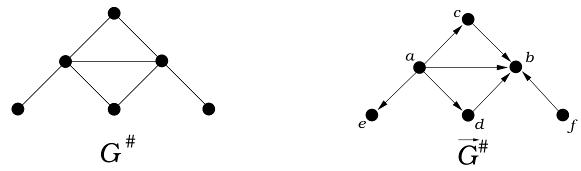

Figure 2:

Proposition 1 *Let G be a comparability graph, u, v different vertices in $V(G)$. Then u is dominated by v if and only if for every partial order $\leq$ in $V(G)$ associated to G, u is above or below dominated by v in $(V, \leq)$.*

In figure 2, we present a Helly comparability graph $G^{\#}$ and we assign a direction to its edges according to an associated order $\leq$. Observe that this graph admits only two associated orders, each one reversal of the other. Notice that the vertices of $G^{\#}$ verify that: c and d are both dominated by a, c and d are both dominated by b, e is dominated only by a, f is dominated only by b. Observe that in the poset $(V(G^{\#}), \leq)$ we have c and d below dominated by a

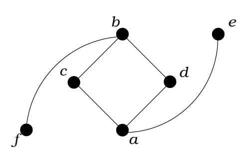

Figure 3: Diagram for the poset $(V(G^{\#}), \leq)$

and above dominated by b. Vertex e is below dominated by a and f is above dominated by b.

Theorem 4 $K^{-2}(G^{\#}) \cap Helly \cap Comp = \emptyset$

Proof: Let H be a reduced comparability graph in $K^{-2}(G^{\#})$. Since $G^{\#}$ is an induced subgraph of H, each associated order of H induces an associated order of $G^{\#}$. Call $\leq$ an order of H which induces the order considered on $G^{\#}$. Since, by Proposition 1 and Theorem 3, no vertex in $(V(G^{\#}), \leq)$ is above/below dominated by a different vertex in $(V(H), \leq)$, it follows that (b, c, a, d) is a cycle in the diagram of $(V(H), \leq)$ satisfying that b and c are not above dominated by a same vertex, and c and a are not below dominated by a same vertex. Therefore $(V(H), \leq)$ has the 2-GN configuration and H is not Helly.

□

Corollary 1 $K^2(Comp \cap Helly) \neq Comp \cap Helly$

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References

[1] Bandelt, H. J.; Prisner, E., *Clique Graphs and Helly Graphs*, J. Combin. Theory B, 51, (1991) 34–45.

[2] Brandstadt, A.; Dragan, F. F.; Chepoi, V. D.; Voloshim, V. I., *Dually chordal Graphs*, Proccedigs of the 19th. International Workshop on Graph Theoretic Concepts in Computer Science, Lecture Notes in Computer Science, 790, (1994), 237–251.

[3] Escalante, F., *Über Iterierte Clique-Graphen*, Abh. Math. Sem. Univ. Hamburg, 39, (1973), 59–68.

[4] Gutierrez, M., *Tree-Clique Graphs*, Estudos e Comunicações do Instituto de Matemática, Universidade Federal do Rio de Janeiro, 63, (1996), 7–26.

[5] Gutierrez, M.; Meidanis, J., *Algebraic Theory for Clique Operator*, Journal of the Brazilian Computer Society, 3(7), (2002), 53-64.

[6] Hedman, B., *Clique graphs of time graphs*, J. of Combin. Theory, 3, (1984), 270–278.

[7] Monma, C. L.; Wei, V. K., *Intersection Graphs of Paths in a Tree*, J. Combin. Theory B, 41, (1986), 141–181.

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