RECOGNIZING SELF-CLIQUE GRAPHS

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Abstract

The *clique graph* $K(G)$ of a graph G is the intersection graph of all the (maximal) cliques of G. A connected graph G is *self-clique* if $G \cong K(G)$. Self-clique graphs have been studied since 1973. We proposed recently a hierarchy of self-clique graphs: Type 3 $\subsetneq$ Type 2 $\subsetneq$ Type 1 $\subsetneq$ Type 0. Here we study the computational complexity of the corresponding recognition problems. We show that recognizing graphs of Type 0 and Type 1 is polynomially equivalent to the graph isomorphism problem. Partial results for Types 2 and 3 are also presented.

1 Preliminaries

Self-clique graphs, discovered by Escalante in [7], have also been studied in [1, 4, 6, 11–13]. Hedman [10] asked if such graphs can be characterized. We refer to [15] for the bibliography on clique graphs. We learned recently that Balconi [2] also has related results. Our few undefined terms and symbols are standard and can be found in [5, 8, 9].

If G is a (finite, simple) graph and $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X, and we usually identify X with $G[X]$. In particular we often write $x \in G$ instead of $x \in V(G)$, and identify the cliques of G (which are maximal complete subgraphs) with their vertex sets.

We denote the distance between two vertices $x, y \in G$ by $d(x, y)$ or $d_G(x, y)$. The disk of radius r centered at x in G is denoted by $D^r_G(x) = \{y \in G : d(x, y) \le r\}$. When $r = 1$, $D^1_G(x) = N_G[x]$ is the *closed neighbourhood* of x. On the other hand, the *neighbourhood* $N_G(x)$ is the set of all neighbours of x in G.

Keywords: clique graphs, self-clique graphs, vertex-clique bipartite graph, computational complexity, graph isomorphism problem.

*Partially supported by CONACyT, Grant 400333-5-27968E.

We say that a vertex $v \in G$ is *dominated (by w)* if $N_G[v] \subseteq N_G[w]$ for some $w \neq v$ in G. For instance, in a triangleless graph, dominated means terminal. The *n-th power graph* G^n has $V(G^n) = V(G)$, $E(G^n) = \{\{x, y\} : d_G(x, y) \leq n\}$.

A family $\mathcal{F}$ of subsets of a set $X \neq \varnothing$ is *Helly* if $\cap\mathcal{S} \neq \varnothing$ for any pairwise intersecting subfamily $\mathcal{S} \subseteq \mathcal{F}$. A graph G is *Helly* if the family of cliques of G is Helly. For instance, every triangleless graph is Helly.

The *vertex-clique bipartite graph* (see [18]) $BK(G)$ of G has $V(BK(G)) = V(G) \cup V(K(G))$ and $E(BK(G)) = \{\{x, Q\} : x \in Q\}$. The neighbourhoods in $BK(G)$ are as follows: $N(Q) = Q \subseteq V(G)$ for $Q \in K(G)$ and $N(v) = v^* \subseteq V(K(G))$ for $v \in G$. Here $v^* = \{Q \in K(G) : v \in Q\}$ is the *star* of v.

Let's recall the hierarchy of self-clique graphs studied in [11]. A graph G is of *Type 0* if it is self-clique: connected and $G \cong K(G)$. A graph G is of *Type 1* if it is a Helly self-clique graph. The distinction between Helly and non-Helly self-clique graphs was already made by Escalante in [7]. A connected graph G is *involutive* or of *Type 2* if $B = BK(G)$ has a part-switching involution, that is, B has an automorphism $\varphi : B \to B$ such that $\varphi(V(G)) = V(K(G))$, $\varphi(V(K(G))) = V(G)$ and $\varphi^2 = \mathrm{id}$. It was shown in [11] that all previously published graphs of Type 1 were indeed of Type 2. Finally, a connected graph G is said to be *clique-disk* or of *Type 3* if G does not have dominated vertices and there is a graph R such that $G = R^2$ and the cliques of G are precisely the disks of radius 1 of R, in symbols: $V(K(G)) = \dot{\bigcup}_{x \in G}\{N_R[x]\}$.

In this paper we are interested in the time complexity of recognizing whether a given graph G is of Type N for N = 0, 1, 2, 3. We shall use the following tags for the indicated decision problems:

- ISO: Graph isomorphism problem.
- SELF: Self-clique graph recognition.
- HSELF: Helly self-clique graph recognition.
- INVO: Involutive graph recognition.
- CDISK: Clique-disk graph recognition.

Our graphs are usually loopless, but for auxiliary purposes we also use *possibly loopy* graphs (always called H) that are allowed to have at most one loop at each vertex. Notice that under these circumstances, $x \in N_H(x)$ iff H has a loop at x. For such a possibly loopy graph we define the *strict square* $H^{[2]}$ as the (loopless) graph that has the same vertex set as H and in which two vertices x, y are adjacent iff they can be joined by two distinct edges $\{x, u\}$ and

$\{u, y\}$ of H (here a loop counts as an edge).

We say that a possibly loopy graph H is *good* iff the family of neighbourhoods $\{N_H(x) : x \in H\}$ is Helly and no neighbourhood is contained in another one: $N_H(x) \subseteq N_H(y) \Rightarrow x = y$. We shall use the following theorems proved in [11]:

Theorem 1.1 [11] *$BK(G)$ is good if and only if G is Helly without dominated vertices.*

Theorem 1.2 [11] *A graph G is involutive if and only if $G \cong H^{[2]}$ for some possibly loopy, good, connected, non-bipartite graph H.*

Theorem 1.3 (The Hierarchy Theorem [11]) *The following proper containment relations among the classes of self-clique graphs hold:*

$$\textit{Type 3} \subsetneq \textit{Type 2} \subsetneq \textit{Type 1} \subsetneq \textit{Type 0}$$

2 Self-Clique Graphs

Let G be a graph, with p vertices, q edges and μ maximal independent sets. Tsukiyama, Ide, Ariyoshi and Shirakawa [17] presented an algorithm (which we shall call the TIAS algorithm) that can compute all the maximal independent sets of G in $O(pq\mu)$ time. Indeed this algorithm computes a new maximal independent set within every $O(pq)$ time interval.

Since we can complement a graph in $O(p^2)$ time, it follows that we can compute a polynomial number of cliques in polynomial time. In particular, given a graph G we can determine if it has exactly $|G|$ cliques (and compute them) in $O(p^2(p^2 - q))$ time. Thus, in order to decide whether G is self-clique or not, we can compute $K(G)$ (or stop with answer "no" if $|K(G)| \neq |G|$) in polynomial time and then apply an isomorphism test. It follows that SELF is polynomially reducible to ISO. Since we know by Szwarcfiter [16] that Hellyness is polynomially verifiable, it is clear that HSELF is also polynomially reducible to ISO. We shall see here that the converses also hold.

We *subdivide* a graph G by replacing each edge by a new path of length 2. If $\widetilde{G}$ is the subdivision of G, then $\widetilde{G}$ is bipartite and has a natural bipartition $\{X, Y\} = \{\text{old vertices, new vertices}\}$. If G is connected so is $\widetilde{G}$ and its bipartition is unique, so given $\widetilde{G}$ and the fact that the part X contains an old vertex (hence all) one recovers G by $G = \widetilde{G}^{[2]}[X]$. Note that, since every new vertex in $\widetilde{G}$ has degree 2, whenever G is connected and not a cycle it is quite easy to see which part contains the old vertices.

Let G_1 and G_2 be any two disjoint graphs. Take G_1 and add three extra vertices $\{x_1, y_1, z_1\}$, make x_1 adjacent to every vertex in $G_1 \cup \{y_1, z_1\}$ and make y_1 adjacent to every vertex in $G_1 \cup \{x_1, z_1\}$. Call the resulting graph G_1'. Now subdivide G_1' to obtain G_1''. Do the same to G_2 with three other extra vertices $\{x_2, y_2, z_2\}$ to obtain G_2' and then subdivide to get G_2''. Then G_1'' and G_2'' are connected, triangleless (therefore Helly) and without dominated (i.e. terminal) vertices. We also have that G_1'' and G_2'' are isomorphic iff G_1 and G_2 are so: Indeed, the only maximal-degree vertices in G_i'' are the extra vertices x_i and y_i, so any isomorphism $G_1'' \to G_2''$ induces an isomorphism $G_1' \to G_2'$ and so $G_1 \cong G_2$.

Now define a new graph G_{12} by $V(G_{12}) = V(G_1'') \cup V(K(G_2''))$ and $E(G_{12}) = E(G_1'') \cup E(K(G_2'')) \cup \{\{z_1, Q\} : Q \in K(G_2'')$ and $z_2 \in Q\}$. This is just the disjoint union of G_1'' and $K(G_2'')$ plus 2 specific edges.

Theorem 2.1 *Given any two graphs G_1 and G_2, construct G_{12} as above. Then the following conditions are equivalent:*

1. *G_1 and G_2 are isomorphic.*
2. *G_{12} is involutive.*
3. *G_{12} is Helly self-clique.*
4. *G_{12} is self-clique.*

Proof: **(1)⇒(2):** If $G_1 \cong G_2$, there is an isomorphism $\tau : G_1'' \to G_2''$ satisfying $\tau(z_1) = z_2$. Then $\tau_K : K(G_1'') \to K(G_2'')$, defined by $\tau_K(Q) = \{\tau(x) : x \in Q\}$, is also an isomorphism. We know by 1.1 that $BK(G_1'')$ is good. Now attach a loop at z_1 to $BK(G_1'')$ to obtain H. It is easy to check that H is still good, and it is clearly connected and non-bipartite. Since $H^{[2]} \cong G_{12}$ via the isomorphism defined by $\varphi(x) = x$ for $x \in G_1''$ and $\varphi(Q) = \tau_K(Q)$ for $Q \in K(G_1'')$, G_{12} is involutive by 1.2.

(2)⇒ (3) ⇒ (4): This follows from the Hierarchy Theorem 1.3.

(4)⇒(1): Define G_{21} by $V(G_{21}) = V(G_2'') \cup V(K(G_1''))$ and $E(G_{21}) = E(G_2'') \cup E(K(G_1'')) \cup \{\{z_2, Q\} : Q \in K(G_1'')$ and $z_1 \in Q\}$. It is a routine verification to check that $G_{21} \cong K(G_{12})$ via the isomorphism defined by $\varphi(z_2) = \{Q \in K(G_2'') : z_2 \in Q\} \cup \{z_1\}$, $\varphi(x) = \{Q \in K(G_2'') : x \in Q\}$ for $x \neq z_2$, $x \in G_2'' \subseteq G_{21}$ and $\varphi(Q) = Q$ for $Q \in K(G_1'') \subseteq G_{21}$.

Now, assuming that $G_{12} \cong K(G_{12})$, there is an isomorphism $\tau : G_{12} \to G_{21}$. By construction, G_1'' and G_2'' do not have cutpoints. Since the cliques of G_i'' are its edges, also $K(G_1'')$ and $K(G_2'')$ are cutpoint-free. Then z_1 (resp. z_2) is the only cutpoint of G_{12} (resp. G_{21}). Now $\tau(z_1) = z_2$, so $G_1'' \subseteq G_{12}$ must be mapped by τ onto $G_2'' \subseteq G_{21}$ or onto $K(G_1'') \cup \{z_2\} \subseteq G_{21}$. Since G_1'' and G_2'' are triangleless but $K(G_1'') \cup \{z_2\}$ is not, $\tau(G_1'') = G_2''$. Thus G_1'' and G_2'' are isomorphic, and so are G_1 and G_2.

□

Since G_2'' has $|E(G_2'')| = 2|E(G_2)| + 4|V(G_2)| + 6$ cliques, we can construct $K(G_2'')$ and hence G_{12} in polynomial time. Then we have proved the following:

Theorem 2.2 *ISO is polynomially reducible to SELF, HSELF and INVO. Furthermore, SELF and HSELF are polynomially equivalent to ISO.*

The authors of [4] have recently informed us that they also independently proved that ISO and SELF are polynomially equivalent.

Problem 2.3 *Determine the time complexity of INVO and CDISK.*

3 Clique-Disk Graphs

By the previous section we only know that INVO is (up to a polynomial transformation) at least as difficult as ISO. But we know even less about the clique-disk recognition problem: We know nothing, apart from the obvious CDISK $\in \mathcal{NP}$. Motwani and Sudan [14] showed that computing square roots of graphs is $\mathcal{NP}$-hard, which seems to suggest that CDISK could be $\mathcal{NP}$-complete. However, all the graphs constructed by Motwani and Sudan in their proof have exponentially many cliques, so those graphs are "highly non self-clique", very far from our domain.

In [4], Bondy, Durán, Lin and Szwarcfiter introduced an important and large subclass of Type 3 (which indeed motivated the definition of Type 3 in [11]). The purpose of this section is to prove that the graphs in this subclass (which we shall call BDLS graphs) are recognizable in polynomial time.

A connected graph G is a *BDLS graph* if $G = R^{2k}$ for some graph R with $\delta(R) \geq 2$, $g(R) \geq 6k+1$ and $k \geq 1$. Here $g(R)$ is the girth of R.

Theorem 3.1 *Let G be a graph. For each vertex $x \in G$ define recursively the sets $F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots$ by:*

$$\begin{aligned} F_0(x) &= x^* = \{Q \in K(G) : x \in Q\} \\ F_j(x) &= \left\{Q \in F_{j-1}(x) : Q \subseteq \bigcup (F_{j-1}(x) \backslash \{Q\})\right\}. \end{aligned}$$

If $G = R^{2k}$ is a BDLS graph, then for all $j \geq 0$ and $x \in G$ we have

$$F_j(x) = \{D_R^k(y) : y \in D_R^{k-j}(x)\}.$$

Thus: $F_{k-1}(x) = \{D_R^k(y) : y \in N_R[x]\}$, $F_k(x) = \{D_R^k(x)\}$ and $F_{k+1}(x) = \varnothing$.

Proof: Let $G = R^{2k}$ be a BDLS graph. Recall from [4] (see also [3, 11]) that: The cliques of G are precisely the disks of radius k of R, the rule $x \mapsto D_R^k(x)$ is an isomorphism from G to $K(G)$ and each $D_R^k(x)$ induces in R a tree of radius k with all the leaves at distance k from the center x.

Since $x \in D_R^k(y)$ if and only if $y \in D_R^k(x)$, we have $F_0(x) = \{D_R^k(y) : y \in D_R^k(x)\}$ as required for $j = 0$.

By induction, assume that $F_j(x) = \{D_R^k(y) : y \in D_R^{k-j}(x)\}$ for some j.

The set $D_R^{k-j}(x)$ induces a tree T_x in R, and a vertex $y \in R$ is a leaf of T_x if and only if $d_R(y,x) = k-j$. Now $y \in D_R^{k-j-1}(x) \Leftrightarrow N_R[y] \subseteq T_x \Leftrightarrow D_R^k(y) \subseteq \bigcup\{D_R^k(z) : z \in N_R[y] \cap T_x \text{ , } z \neq y\} \Leftrightarrow D_R^k(y) \in F_{j+1}(x)$.

□

Therefore, if $G = R^{2k}$ is a BDLS graph, R and k are determined by G. Indeed: k is the number for which $|F_k(x)| = 1$ for all (or just one) $x \in G$ and we can reconstruct R by $V(R) = V(G)$ and $\{x,y\} \in E(R)$ iff $x \neq y$ and $F_k(y) \subseteq F_{k-1}(x)$.

Now assume we want to determine whether a graph G is a BDLS graph. Thanks to the TIAS algorithm [17], we can construct each $F_0(x)$ in polynomial time (or determine that G does not have exactly $|V(G)|$ cliques, thus answering "no" and stopping computation). Then, as described above, we can also reconstruct k and R (or determine that there are no such k and R) in polynomial time: Since we always have $F_j(x) = F_{j+1}(x)$ for some $j \leq |V(G)|$ we only have to compute (at worst) $|V(G)|^2$ of the $F_j(x)$'s. Finally, we just have to check that

$G = R^{2k}$ (equality, not isomorphism!) $\delta(R) \geq 2$, $g(R) \geq 6k+1$ and that R is connected. It is clear that all these operations can be carried out in polynomial time, so we have proved:

Theorem 3.2 *BDLS graphs are recognizable in polynomial time.*

4 Final Remarks

Given two graphs A and B, the *strong product* $A \boxtimes B$ is the loopless graph with vertex set $V(A \boxtimes B) = V(A) \times V(B)$ where $\{(a_1, b_1), (a_2, b_2)\} \in E(A \boxtimes B)$ iff a_1 and a_2 are adjacent or equal AND b_1 and b_2 are adjacent or equal.

Now, take $m, n \geq 7$ and $P = C_n \boxtimes C_m$ (here C_n is a cycle of length n). A direct verification shows that $G = P^2$ satisfies $K(G) = \{N_P[v] : v \in P\}$, so it is clique-disk. If we try our BDLS graph recognizing algorithm on this one, we get that for all $v \in G$:

$$\begin{aligned} F_0(v) &= \{N_P[v+\alpha] : \alpha \in \{-1,0,1\} \times \{-1,0,1\}\}, \\ F_1(v) &= \{N_P[v+\alpha] : \alpha \in \{(0,1),(0,-1),(0,0),(1,0),(-1,0)\}\} \text{ and} \\ F_2(v) &= \{N_P[v]\}. \end{aligned}$$

Then we define R by $V(R) = V(G) = V(P)$ and $\{u, v\} \in E(R)$ if and only if $u - v \in \{(0,1),(0,-1),(1,0),(-1,0)\}$. Since k should be 2, we observe that $\delta(R) = 4 \geq 2$, but $g(R) = 4 < 6k+1 = 13$ and $G \neq R^4$.

We conclude that the BDLS class is properly contained in Type 3, and that the final verifications in our algorithm are not superfluous (at least these two: $g(R) \geq 6k+1$ and $G = R^{2k}$).

On the other hand we note that, in this case, computing $F_2(v) = \{N_P[v]\}$ gives us the isomorphism $v \leftrightarrow N_P[v]$ between G and $K(G)$. If this were always the case for a clique-disk graph, we would have a polynomial time algorithm for CDISK. Unfortunately this is not so, since the clique-disk graph $G = (R_8)^2$ (see Fig. 1) has

$$\begin{aligned} F_0(a_i) &= \{N_{R_8}[v] : v \in \{a_{i-1}, a_i, a_{i+1}, x_{i-1}, x_i, b_i\}\}, \\ F_1(a_i) &= \{N_{R_8}[v] : v \in \{a_i, x_{i-1}, x_i, b_i\}\}, \\ F_2(a_i) &= \{N_{R_8}[a_i], N_{R_8}[b_i]\} \text{ and} \\ F_3(a_i) &= \varnothing = F_4(a_i) = F_5(a_i) = \cdots \end{aligned}$$

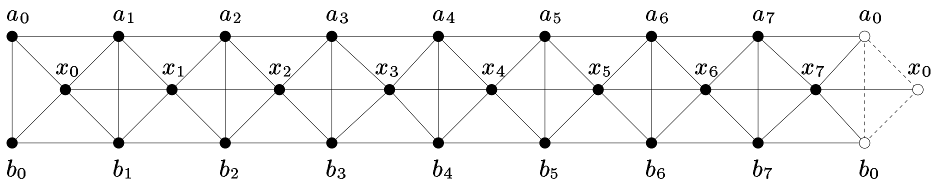

Figure 1: The graph R_8 (identify vertices with same labels).

Acknowlegments. The authors thank the referee who pointed out a mistake.

References

[1] Balakrishnan, R.; Paulraja, P., *Self-clique graphs and diameters of iterated clique graphs*, Utilitas Math. 29 (1986) 263–268.

[2] Balconi, G., *Matrix characterizations of self-dual graphs*, Istit. Lombardo Accad. Sci. Lett. Rend. A 113 (1979) 360–365 (1981).

[3] Bondy, A.; Durán, G.; Lin, M. C.; Szwarcfiter, J. L., *On self-clique graphs and permutation matrices*, Submitted. Available from: www.dc.uba.ar/people/profesores/willy/publicaciones.html.

[4] Bondy, A.; Durán, G.; Lin, M. C.; Szwarcfiter, J. L., *A sufficient condition for self-clique graphs*, Electronic Notes on Discrete Mathematics 7 (2001) 19–23.

[5] Bondy, J. A.; Murty, U. S. R., *Graph theory with applications*, American Elsevier Publishing Co., Inc., New York, 1976.

[6] Chia, G. L., *On self-clique graphs with given clique sizes*, Discrete Math. 212 (2000) 185–189. Combinatorics and applications (Tianjin, 1996).

[7] Escalante, F., *Über iterierte Clique-Graphen*, Abh. Math. Sem. Univ. Hamburg 39 (1973) 59–68.

[8] Garey, M. R.; Johnson, D. S., *Computers and intractability*, W. H. Freeman and Co., San Francisco, Calif., 1979. A guide to the theory of NP-completeness, A Series of Books in the Mathematical Sciences.

[9] Harary, F., *Graph theory*, Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London, 1969.

[10] Hedman, B., *Diameters of iterated clique graphs*, Hadronic J. 9 (1986) 273–276.

[11] Larrión, F.; Neumann-Lara, V.; Pizaña, M. A.; Porter, T. D., *A hierarchy of self-clique graphs*, Submitted.

[12] Larrión, F.; Neumann-Lara, V.; Pizaña, M. A.; Porter, T. D., *Self clique graphs with prescribed clique-sizes*, Submitted.

[13] Lim, C. K.; Peng, Y. H., *On graphs without multicliqual edges*, J. Graph Theory 5 (1981) 443–451.

[14] Motwani, R.; Sudan, M., *Computing roots of graphs is hard*, Discrete Appl. Math. 54 (1994) 81–88.

[15] Szwarcfiter, J. L., *A survey on clique graphs*, In Recent Advances in Algorithms and Combinatorics. C. Linhares and B. Reed, eds., Springer-Verlag. To appear.

[16] Szwarcfiter, J. L., *Recognizing clique-Helly graphs*, Ars Combin. 45 (1997) 29–32.

[17] Tsukiyama, S.; Ide, M.; Ariyoshi, H.; Shirakawa, I., *A new algorithm for generating all the maximal independent sets*, SIAM J. Comput. 6 (1977) 505–517.

[18] Wallis, W. D.; Wu, J. L., *Squares, clique graphs and chordality*, J. Graph Theory 20 (1995) 37–45.

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